Perturbations of diagonal matrices by band random matrices

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PERTURBATIONS OF DIAGONAL MATRICES BY BAND RANDOM MATRICES

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Abstract. We exhibit an explicit formula for the spectral density of a (large) random matrix which is a diagonal matrix whose spectral density converges, perturbated by the addition of a symmetric matrix with Gaussian entries and a given (small) limiting variance profile.

1. Perturbation of the spectral density of a large diagonal matrix

In this paper, we consider the spectral measure of a random matrix D_n^ε defined by $D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n$, for D_n a deterministic diagonal matrix whose spectral measure converges and X_n an Hermitian or real symmetric matrix whose entries are Gaussian independent variables, with a limiting variance profile (such matrices are called *band matrices*). We give a first order Taylor expansion, as $\varepsilon \to 0$, of the limit spectral density, as $n \to \infty$, of D_n^ε.

The proof is elementary and based on a formula given in [12] for the Cauchy transform of the limit spectral distribution of D_n^ε as $n \to \infty$.

For each n, we consider an Hermitian or real symmetric random matrix $X_n = [x_{i,j}^n]_{i,j=1}^n$ and a real diagonal matrix $D_n = \operatorname{diag}(a_n(1), \ldots, a_n(n))$. We suppose that:

(a) the entries $x_{i,j}^n$ of X_n are independent (up to symmetry), centered, Gaussian with variance denoted by $\sigma_n^2(i,j)$,

(b) for a certain bounded function σ defined on $[0,1]\times[0,1]$ and a certain bounded real function f defined on $[0,1]$, we have, in the L^∞ topology,
$$\sigma_n^2(\lfloor nx\rfloor, \lfloor ny\rfloor) \underset{n\to\infty}{\longrightarrow} \sigma^2(x,y) \qquad \text{and} \qquad a_n(\lfloor nx\rfloor) \underset{n\to\infty}{\longrightarrow} f(x),$$

(c) the set of discontinuities of the function σ is closed and intersects a finite number of times any vertical line of the square $[0,1]^2$.

For $\varepsilon \geq 0$, let us define, for all n,
$$D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n.$$

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It is known, from Shlyakhtenko in [12, Th. 4.3] (see also [2], which also provides a fluctuation result), that as n tends to infinity, the spectral distribution of D_n^ε tends to a limit μ_ε with Cauchy transform

$$C_\varepsilon(z) = \int_{x=0}^1 C_\varepsilon(x,z)\mathrm{d}x,$$

where $C_\varepsilon(x,\,\cdot\,)$ is defined by the fact that it is analytic, maps the upper half-plane $\mathbb{C}^+$ into the lower one $\mathbb{C}^-$, and satisfies the relation

$$C_\varepsilon(x,z) = \frac{1}{z - f(x) - \varepsilon\int_{y=0}^1 \sigma^2(x,y)C_\varepsilon(y,z)\mathrm{d}y}. \tag{1}$$

Our goal here is to understand $\mu_\varepsilon - \mu$ for small values of ε. Let us introduce the set $\mathcal{T}$ of test functions we shall use here. We define

$$\mathcal{T} = \left\{ t \longmapsto \frac{1}{z-t}\,;\, z \in \mathbb{C}^+ \right\}.$$

Let us now define the *Hilbert transform*, denoted by $H[u]$, of a function u:

$$H[u](s) := \mathrm{p.\,v.} \int_{t\in\mathbb{R}} \frac{u(t)}{s-t}\mathrm{d}t = \int_{y\in\mathbb{R}} \frac{u(s-y)-u(s)}{y}\mathrm{d}y.$$

Before stating our main result, let us make some assumptions on the functions σ and f:

(d) the push-forward μ of the uniform measure on $[0,1]$ by the function f has a density ρ with respect to the Lebesgue measure on $\mathbb{R}$,

(e) there exists a symmetric function $\tau(\,\cdot\,,\,\cdot\,)$ such that for all x, y, $\sigma^2(x,y) = \tau(f(x), f(y))$,

(f) there exist $\eta_0 > 0, \alpha > 0$ and $C < \infty$ such that for almost all $s \in \mathbb{R}$, for all $t \in [s-\eta_0, s+\eta_0]$, $\quad |\tau(s,t)\rho(t) - \tau(s,s)\rho(s)| \le C|t-s|^\alpha$.

Note that by hypothesis (f) and by the boundedness of the function f, the function

$$s \longmapsto \rho(s)H[\tau(s,\,\cdot\,)\rho(\,\cdot\,)](s)$$

is well defined and compactly supported.

Theorem 1. *Under the hypotheses (a) to (f), as $\varepsilon \to 0$, for all $g \in \mathcal{T}$,*

$$\int g(s)\mathrm{d}\mu_\varepsilon(s) = \int g(s)\mathrm{d}\mu(s) - \varepsilon\int g'(s)F(s)\mathrm{d}s + o(\varepsilon),$$

with $F(s) := -\rho(s)H[\tau(s,\,\cdot\,)\rho(\,\cdot\,)](s)$.

As a consequence, if the function $F(\,\cdot\,)$ has bounded variations, then

$$\mu_\varepsilon = \mu + \varepsilon\mathrm{d}F + o(\varepsilon).$$

Remark 1. *Roughly speaking, this theorem states that*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{\text{spectral law}(D_n^\varepsilon) \;-\; \text{spectral law}(D_n)}{\varepsilon} = \mathrm{d}F.$$

It would be interesting to let ε and n tend to 0 and ∞ together, and to find out the adequate rate of convergence to get a deterministic limit or non degenerated fluctuations. We are working on this question.

Remark 2. *This result provides an analogue, for our random matrix model, of the following formula about real random variables (valid when Y is centered and independent of X):*

$$\mathrm{density}_{X+\sqrt{\varepsilon}Y}(s) = \mathrm{density}_X(s) + \varepsilon\frac{\mathbb{E}[Y^2]}{2}\,\mathrm{density}''_X(s) + o(\varepsilon).$$

Remark 3. *In the case where X_n is a GUE or GOE matrix, the limiting spectral distribution of D_n^ε as $n \to \infty$ is the free convolution of the limiting spectral distribution of D_n with a semi-circle distribution. Several papers are devoted to the study of qualitative properties (like regularity) of the free convolution (see [8, 7, 4, 3, 6]). Besides, it has recently been proved that type-B free probability theory allows to give Taylor expansions, for small values of t, of the moments of $\mu_t \boxplus \nu_t$ for two time-depending probability measures μ_t and ν_t (see [5, 10, 9]). Our work differs from the ones mentioned above by the fact that we allow to perturb D_n by any band matrix, but also by the fact that it is focused on the density and not on the moments, giving an explicit formula rather than qualitative properties.*

Proof. For all $z \in \mathbb{C}^+$, we have

$$|C_\varepsilon(x,z)| \le \frac{1}{\Im z}. \tag{2}$$

Indeed, for all y, z such that $z \in \mathbb{C}^+$, $C_\varepsilon(y,z) \in \mathbb{C}^-$. As a consequence, the imaginary part of the denominator of the right hand term of (1) is larger than $\Im(z)$.

Hence by (1) and (2), as $\varepsilon \to 0$, $C_\varepsilon(x,z) \longrightarrow \frac{1}{z-f(x)}$ uniformly in x.

From what precedes,

$$\begin{aligned}
C_\varepsilon(x,z) - \frac{1}{z-f(x)} &= \frac{\varepsilon\int_{y=0}^1 \sigma^2(x,y)C_\varepsilon(y,z)\mathrm{d}y}{(z-f(x)-\varepsilon\int_{y=0}^1 \sigma^2(x,y)C_\varepsilon(y,z)\mathrm{d}y)(z-f(x))}\\
&= \varepsilon\frac{1}{(z-f(x))^2}\int_{y=0}^1 \sigma^2(x,y)C_\varepsilon(y,z)\mathrm{d}y + o(\varepsilon)\\
&= \varepsilon\frac{1}{(z-f(x))^2}\int_{y=0}^1 \frac{\sigma^2(x,y)}{z-f(y)}\mathrm{d}y + o(\varepsilon)
\end{aligned}$$

where each $o(\varepsilon)$ is uniform in $x \in [0,1]$.

But for all $a \ne b$, $\frac{1}{(z-a)^2(z-b)} = \frac{1}{(a-b)^2}\left(\frac{1}{z-b} - \frac{1}{z-a} - \frac{b-a}{(z-a)^2}\right)$, hence since the Lebesgue measure of the set $\{y \in [0,1]\,;\, f(y) = f(x)\}$ is null, we have

$$\frac{1}{(z-f(x))^2}\int_{y=0}^1 \frac{\sigma^2(x,y)}{z-f(y)}\mathrm{d}y = \int_{y=0}^1 \frac{\sigma^2(x,y)}{(f(x)-f(y))^2}\left(\frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2}\right)\mathrm{d}y.$$

As a consequence, it follows by an integration in $x \in [0,1]$ that

$$C_\varepsilon(z) - C(z) = \varepsilon\int_{x=0}^1\int_{y=0}^1 \frac{\sigma^2(x,y)}{(f(x)-f(y))^2}\left(\frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2}\right)\mathrm{d}y\mathrm{d}x + o(\varepsilon),$$

where $C(\cdot)$ is the Cauchy transform of μ.

Let us now recall that the push-forward of the uniform law on $[0,1]$ by f is the measure $\rho(x)\mathrm{d}x$ and that $\sigma^2(x,y)$ can be rewritten $\sigma^2(x,y) = \tau(f(x), f(y))$. Hence

$$C_\varepsilon(z) - C(z) = \varepsilon\int_{s\in\mathbb{R}}\int_{t\in\mathbb{R}}\left\{\frac{1}{z-t} - \frac{1}{z-s} - \frac{1}{(z-s)^2}(t-s)\right\}\frac{\tau(s,t)}{(s-t)^2}\rho(s)\rho(t)\mathrm{d}t\mathrm{d}s + o(\varepsilon).$$

This allows us to write that for any test function $g \in \mathcal{T}$,

$$\lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(g) - \mu(g)}{\varepsilon} = \Lambda(g),$$

where

$$\Lambda(g) = \int_{(s,t)\in\mathbb{R}^2} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \mathrm{d}t\mathrm{d}s.$$

Note that by the Taylor-Lagrange formula, for all s, t,

$$\left| \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \right| \le \frac{\rho(s)\rho(t) \times \|\tau(\,\cdot\,,\,\cdot\,)\|_{L^\infty} \|g''\|_{L^\infty}}{2},$$

so that, since ρ is a density, by dominated convergence,

$$\Lambda(g) = \lim_{\eta\to 0} \int_{\substack{(s,t)\in\mathbb{R}^2 \\ |s-t|>\eta}} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \mathrm{d}s\mathrm{d}t.$$

But by symmetry, for all $\eta > 0$,

$$\int_{\substack{(s,t)\in\mathbb{R}^2 \\ |s-t|>\eta}} \{g(t) - g(s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \mathrm{d}s\mathrm{d}t = 0.$$

As a consequence, $\Lambda(g) = \lim_{\eta\to 0} \Lambda_\eta(g)$, with

$$\Lambda_\eta(g) := \int_{\substack{(s,t)\in\mathbb{R}^2 \\ |s-t|>\eta}} g'(s) \frac{\tau(s,t)}{s-t} \rho(s)\rho(t) \mathrm{d}s\mathrm{d}t.$$

Let us prove that almost all $s \in \mathbb{R}$, $\lim_{\eta\to 0} \int_{\substack{t\in\mathbb{R} \\ |s-t|>\eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} \mathrm{d}t$ exists and that

$$\Lambda(g) = \int_{s\in\mathbb{R}} g'(s) \left(\lim_{\eta\to 0} \int_{\substack{t\in\mathbb{R} \\ |s-t|>\eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} \mathrm{d}t \right) \mathrm{d}s.$$

For $\eta > 0$ and $s \in \mathbb{R}$, set

$$\theta_\eta(s) := \int_{\substack{t\in\mathbb{R} \\ |s-t|>\eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} \mathrm{d}t.$$

Set also $M := \|f\|_{L^\infty}$. Then the support of the function ρ is contained in $[-M, M]$, and so does the support of the function θ_η, for any $\eta > 0$. For almost all $s \in [-M, M]$, $\lim_{\eta\to 0} \theta_\eta(s)$ exists by the formula

$$\theta_\eta(s) = \int_{t\in[s-2M, s-\eta]\cup[s+\eta, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t) - \tau(s,s)\rho(s)\rho(s)}{s-t} \mathrm{d}t$$

and by Hypothesis (f). Moreover, for η_0 as in Hypothesis (f),

$$\begin{aligned}
|\theta_\eta(s)| &\le 2C\rho(s) \int_{t=s+\eta}^{s+\eta_0} (s-t)^{\alpha-1} \mathrm{d}t + \int_{t\in[s-2M, s-\eta_0]\cup[s+\eta_0, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} \mathrm{d}t \\
&\le \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{1}{\eta_0} \int_{t\in\mathbb{R}} \tau(s,t)\rho(s)\rho(t) \mathrm{d}s\mathrm{d}t \\
&\le \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{\|\tau(\,\cdot\,,\,\cdot\,)\|_{L^\infty}}{\eta_0} \rho(s).
\end{aligned}$$

Hence by dominated convergence, $\int_{s\in\mathbb{R}} g'(s)\lim_{\eta\to 0}\theta_\eta(s)\mathrm{d}s = \lim_{\eta\to 0}\int_{s\in\mathbb{R}} g'(s)\theta_\eta(s)\mathrm{d}s$, i.e.

$$\Lambda(g) = \int_{s\in\mathbb{R}} g'(s)\left(\lim_{\eta\to 0}\int_{\substack{t\in\mathbb{R}\\|s-t|>\eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t}\mathrm{d}t\right)\mathrm{d}s.$$

$\square$

2. Examples

2.1. **Perturbation of a uniform distribution by a standard band matrix.** Let us consider the case where $f(x)=x$ (so that μ is the uniform distribution on $[0,1]$) and $\sigma^2(x,y)=\mathbb{1}_{|y-x|\le\ell}$, where ℓ is a fixed parameter in $[0,1]$ (the width of the band). In this case, $\tau(\,\cdot\,,\,\cdot\,)=\sigma^2(\,\cdot\,,\,\cdot\,)$ and

$$F(s) = \mathbb{1}_{(0,1)}(s)\log\left(\frac{\ell\wedge(1-s)}{\ell\wedge s}\right).$$

For small values of ε and large values of n, the density ρ_ε of the eigenvalue distribution μ_ε of D_n^ε is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon\frac{\partial}{\partial s}F(s) + o(\varepsilon) = \mathbb{1}_{(0,1)}(s) - \varepsilon\left(\frac{\mathbb{1}_{(0,\ell)}(s)}{s} + \frac{\mathbb{1}_{(1-\ell,1)}(s)}{1-s}\right) + o(\varepsilon),$$

which means that the additive perturbation $\sqrt{\frac{\varepsilon}{n}}X_n$ alters the spectrum of D_n essentially by decreasing the amount of extreme eigenvalues. This phenomenon is illustrated by Figure 1 (where we ploted the cumulative distribution functions rather than the densities for visual reasons).

(a) Case where $n=4.10^3$, $\varepsilon=10^{-2}$, with width $\ell=0.2$

(b) Case where $n=4.10^3$, $\varepsilon=10^{-2}$, with width $\ell=0.9$

Figure 1. **Perturbation of a uniform distribution by a standard band matrix:** plot of the functions $F(\,\cdot\,)$ and $\frac{F_{D_n^\varepsilon}(\,\cdot\,)-F_{D_n}(\,\cdot\,)}{\varepsilon}$ (with $F_{D_n^\varepsilon}(\,\cdot\,)$ and $F_{D_n}(\,\cdot\,)$ the cumulative eigenvalue distribution functions of D_n^ε and D_n) for different values of ℓ.

2.2. **Perturbation of the triangular pulse distribution by a GOE matrix.** Let us consider the case where $\rho(x)=(1-|x|)\mathbb{1}_{[-1,1]}(x)$ and $\sigma^2\equiv 1$ (what follows can be adapted to the case $\sigma^2(x,y)=\mathbb{1}_{|y-x|\le\ell}$, but the formulas are a bit heavy). In this case, thanks to the formula (9.6) of $H[\rho(\,\cdot\,)]$ given p. 509 of [11], we get

$$F(s) = (1-|s|)\mathbb{1}_{[-1,1]}(s)\left\{(1-s)\log(1-s)-(1+s)\log(1+s)+2s\log|s|\right\}.$$

For small values of ε and large values of n, the density ρ_ε of the eigenvalue distribution μ_ε of D_n^ε is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon),$$

which implies that the additive perturbation $\sqrt{\frac{\varepsilon}{n}} X_n$ alters the spectrum of D_n by increasing the amount of eigenvalues in $[-1, -0.5] \cup [0.5, 1]$ and decreasing the amount of eigenvalues around zero. This phenomenon is illustrated by Figure 2.

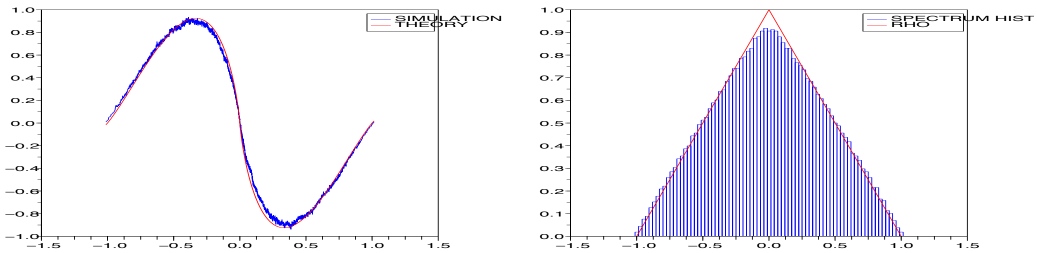

FIGURE 2. **Perturbation of the triangular pulse distribution by a GOE matrix:** *Left:* plot of the functions $F(\,\cdot\,)$ and $\frac{F_{D_n^\varepsilon}(\,\cdot\,) - F_{D_n}(\,\cdot\,)}{\varepsilon}$ (with $F_{D_n^\varepsilon}(\,\cdot\,)$ and $F_{D_n}(\,\cdot\,)$ the cumulative eigenvalue distribution functions of D_n^ε and D_n). *Right:* plot of the eigenvalues histogram of D_n^ε and of the spectral density ρ of D_n. On the right figure, the (infinitesimal) increase of eigenvalues with respect to ρ on $[-1, -0.5] \cup [0.5, 1]$ and the (infinitesimal) decrease around zero can be observed, in agreement with the fact that, as the left figure shows, $F' \gg 0$ on (approximately) $[-1, -0.5] \cup [0.5, 1]$ and $F' \ll 0$ around zero. Both figures were made with the same simulation ($n = 6.10^3$ and $\varepsilon = 10^{-2}$).

2.3. **Free convolution with a semi-circular distribution and complex Burger's equation.** Let us consider the case where $\sigma^2 \equiv 1$, which happens for example if the matrix X_n is taken in the Gaussian Orthogonal Ensemble. In this case, by the theory of free probability developped by Dan Voiculescu (see e.g. [13] or [1, Cor 5.4.11 (ii)]), for all $t \geq 0$,

$$\mu_t = \mu \boxplus \lambda_t,$$

where λ_t is the *semi-circular distribution with variance* t, i.e. the distribution with support $[-2\sqrt{t}, 2\sqrt{t}]$ and density $\frac{1}{2\pi t}\sqrt{4t - x^2}$. In this case, we know by the work of Biane [8, Cor. 2] that for all $t > 0$, μ_t admits a density ρ_t. By the implicit function theorem, and the formula given in [8, Cor. 2], one easily sees that the function $(s, t) \longmapsto \rho_t(s)$ is regular. Then, by Theorem 1 and the fact that the linear span of $\mathcal{T}$ is dense in the set of continuous functions on the real line with null limit at infinity, one easily recovers the following PDE, which is a kind of projection on the real axis of the imaginary part of complex Burger's equation given in [8, Intro.]

$$\begin{cases} \frac{\partial}{\partial t}\rho_t(s) + \frac{\partial}{\partial s}\{\rho_t(s) H[\rho_t(\,\cdot\,)](s)\} = 0, \\ \rho_0(s) = \rho(s). \end{cases} \tag{3}$$

For example, if $\mu = \lambda_c$ for a certain $c > 0$, then by the semi-group property of the semi-circle distribution [1, Ex. 5.3.26], for all $t \geq 0$, $\mu_t = \lambda_{c+t}$ and $\rho_t(s) = \frac{1}{2\pi(c+t)}\sqrt{4(c+t) - s^2}$. One can then verify (3), using the formula (9.21) of $H[\rho_t(\,\cdot\,)]$ given p. 511 of [11].

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