Computing the Worst-Case Peak Gain of Digital Filter in Interval Arithmetic

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Sorbonne Universités, UPMC Univ Paris 06, UMR 7606, LIP6
4, place Jussieu 75005 Paris, France
first_name.last_name@lip6.fr

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The Worst-Case Peak Gain (WCPG) of a Linear Time Invariant (LTI) filter is used to determine the output interval of a filter and in error propagation analysis [5].

Consider a stable LTI filter $\mathcal{H}$ in state-space representation:

$$\mathcal{H}\left\{\begin{array}{rcl}\mathbf{x}(k+1) & = & \mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)\\ \mathbf{y}(k) & = & \mathbf{C}\mathbf{x}(k)+\mathbf{D}\mathbf{u}(k)\end{array}\right. \tag{1}$$

where $\mathbf{u}(k)$ is the input vector, $\mathbf{y}(k)$ is the output vector, $\mathbf{x}(k)$ is the state vector and matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ contain the filter coefficients.

The WCPG of a linear filter can be computed [1] as the infinite sum $\mathbf{W} := |\mathbf{D}| + \sum_{k=0}^{\infty}\left|\mathbf{C}\mathbf{A}^k\mathbf{B}\right|$. In [6] the authors have proposed an algorithm for the reliable evaluation of the WCPG matrix in multiple precision.

However, usually the filter coefficients are rounded prior to implementation, changing $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ by rounding. To provide a reliable filter implementation, these rounding errors must be taken into account in the WCPG computation. We represent the rounded coefficients as interval [2] matrices with small radii. Let $\mathbf{M}^I := \langle\mathbf{M}_c, \mathbf{M}_r\rangle$ to be an interval matrix centered in $\mathbf{M}_c$ with radius $\mathbf{M}_r$. Then, the WCPG matrix of a filter $\mathcal{H} = \left(\mathbf{A}^I, \mathbf{B}^I, \mathbf{C}^I, \mathbf{D}^I\right)$ is an interval $\mathbf{W}^I := \left|\mathbf{D}^I\right| + \sum_{k=0}^{\infty}\left|\mathbf{C}^I\mathbf{A}^{I^k}\mathbf{B}^I\right|$.

In this work we adapt the algorithm presented in [6] to obtain a reliable evaluation of the WCPG interval. The WCPG is computed in two stages: the reliable truncation of the infinite sum and the summation.

We determine the truncation order only for the center matrices but add a correction term after the final step. This step requires to perform an eigenvalue decomposition. To obtain trusted error bounds on the computed eigenvalues we use the Theory of Verified Inclusions developed by S. Rump [4].

The summation is done using Interval Arithmetic in midpoint-radius form. However, powering a dense interval matrix can lead to an interval explosion. Instead of powering $\mathbf{A}^I$ we power an almost diagonal matrix $\mathbf{T}^I$, for which $||\mathbf{T}^I||_2 < 1$ is true. We use an analogue of Gershgorin circle theorem [3] to verify a spectral norm condition that needs to be satisfied for the WCPG sum to converge.

It is obvious that we cannot guarantee an a priori given bound on the WCPG matrix radius $\mathbf{W}_r$ because the radii of the input matrices are the limiting factors. However, when given point coefficient matrices (intervals with zero radii) and an absolute error bound ε we guarantee that the output WCPG interval in not larger than ε in width.

References

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