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Suzuki equations and integrals of motion for supersymmetric CFT

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Abstract

Using equations proposed by J. Suzuki we compute numerically the first three integrals of motion for $N = 1$ supersymmetric CFT. Our computation agrees with the results of ODE-CFT correspondence which was explained in a more general context by S. Lukyanov.

1. Introduction

The present paper contains some preliminary results for a larger project which consists in computing the one-point functions for the supersymmetric sine-Gordon model (ssG) generalising the results of [1,2] obtained for the sine-Gordon case (sG). This problem is interesting because the integrable description of the space of local operators for the ssG model should be derived from that of the inhomogeneous 19-vertex Fateev–Zamolodchikov model while for the sG case it was related to the inhomogeneous 6-vertex model. There is an interesting difference between the two cases: for the 6-vertex case the local observables are created by two fermions while for the 19-vertex case one has to introduce additional Kac–Moody current [3].

The ssG model is a simplest example of integrable model with supersymmetry. The study of ssG model goes back to works [4,5]. The supersymmetry is one of the building blocks of

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modern theoretical physics, which motivated the authors of these papers to try finding solvable two-dimensional examples of supersymmetric models.

The first indispensable step consists in finding the corresponding description in the conformal case like in the paper [6]. The generalisation is already not quite trivial. For example, in the computations of the ground state eigenvalues of the local integrals of motion the paper [6] follows the procedure proposed in [7], namely it uses the NLIE (non-linear integral equations) equations on a half-infinite interval. This allows to develop an analytical procedure for the computation of the eigenvalues in question. Then the procedure is generalised in order to compute the expectation values on a cylinder of the CFT operators in the fermionic basis. Unfortunately, similar procedure for the super CFT case is unknown to us, and we are forced to proceed with numerical computations based on equations which for the 19-vertex model were proposed by J. Suzuki [8]. It should be said that Suzuki equations have been used already for ssG model and its conformal limit in [9].

In the present paper we shall apply the Suzuki equations to the ssG model. In the high temperature limit we compute numerically the eigenvalues of the first three local integrals of motion. We interpolate the results getting exact general formulae. This way of proceeding may look strange having in mind that the formulae in question can be alternatively obtained by the ODE-CFT correspondence [10,11] following Lukyanov [12] as will be explained. However, one should have in mind that we are doing a preliminary work, intending in future to proceed with similar methods to the one-point functions for which not much is known.

The paper is organised as follows. In the second section we give a very brief account of the ssG model viewed as a perturbed CFT. In the third section we give some exposition of the Suzuki equations. This consideration is not original, basically we repeat in appropriate for our goals language the results of the works [8,9]. We simplify our consideration comparing to these papers considering the ground state only. The forth section contains numerical results and their interpolation. Finally, in the last section we explain how the eigenvalues are obtained from ODE-CFT correspondence following [12].

2. Supersymmetric sine-Gordon model

We begin with a very brief description of the supersymmetric sine-Gordon (ssG) field theory, an interested reader can find all necessary details in [13]. In the framework of Perturbed CFT (PCFT) the ssG is considered as a perturbation of the $c = 3/2$ CFT (one boson+one Majorana fermion) by the relevant operator $\Phi = -\mu\bar{\psi}\psi\cos\left(\frac{\beta\varphi}{\sqrt{2}}\right)$:

$$\mathcal{A} = \int\Big(\frac{1}{16\pi}\partial_z\varphi\partial_{\bar{z}}\varphi + \frac{1}{2\pi}\left(\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi}\right) - 2\mu\bar{\psi}\psi\cos\left(\frac{\beta\varphi}{\sqrt{2}}\right)\Big)d^2z\,. \tag{2.1}$$

The dimensional coupling constant μ is of dimension is $[\text{mass}]^{1-\beta^2}$. The scaling dimension of this operator $\Delta_{\text{pert}} = \frac{1}{2}(1+\beta^2)$ is greater than $\frac{1}{2}$, so the UV regularisation is needed. The OPE

$$\Phi(z,\bar{z})\Phi(0) = \frac{1}{(z\bar{z})^{1+\beta^2}} + C\cdot\frac{1}{(z\bar{z})^{1-\beta^2}}\cos\left(\sqrt{2}\beta\varphi\right) + \cdots\,, \tag{2.2}$$

shows that the UV regularisation is simple: the first non-trivial contribution comes with integrable singularity. The model is shown to be integrable, actually this is the simplest example of perturbations of parafermionic models whose integrals of motion are obtained in [14]. The factorisable S-matrix is known, it coincides with the S-matrix for the spin-1 integrable magnetic

[15], in the context of relativistic field theory it was discussed in [16]. The S-matrix is compatible with the $N = 1$ supersymmetry.

The formula for the action (2.1) may contradict the reader's intuition because the supersymmetric classical action contains the additional term $\Phi_1 = -\frac{\pi\mu^2}{\beta^2}\cos\left(\sqrt{2}\beta\varphi\right)$ which we have seen already in the OPE (2.2). In the frame work of the PCFT this term, as it is written, cannot be added to the action for dimensional reasons, at least it needs a new dimensional coupling constant. In the classical limit $\beta \to 0$ the situation becomes more complicated. That is why, when proceeding in the opposite direction, i.e. quantising the classical model by more traditional methods of QFT, one should indeed begin with the supersymmetric action which includes Φ_1 and take care of preserving the supersymmetry. This was done in [17], the result is exactly as expected from our dimensional considerations: the dimensional coupling constants for the two terms of the interaction are renormalised differently, the term with Φ_1 containing vanishing power of the cutoff.

Like in the sine-Gordon case it is often convenient to rewrite the action as

$$\mathcal{A} = \int \Big[\Big(\frac{1}{16\pi}\partial_z\varphi\partial_{\bar z}\varphi + \frac{1}{2\pi}\left(\psi\partial_{\bar z}\psi + \bar\psi\partial_z\bar\psi\right) - \mu\bar\psi\psi e^{-i\frac{\beta}{\sqrt{2}}\varphi}\Big) - \mu\bar\psi\psi e^{i\frac{\beta}{\sqrt{2}}\varphi}\Big] d^2z\,, \quad (2.3)$$

considering the model as perturbation of a supersymmetric CFT with the Virasoro central charge equal to $c = \frac{3}{2}\left(1 - 2(\beta - \beta^{-1})^2\right)$, by the relevant operator (the last term) with scaling dimension $\Delta = \beta^2$.

The mass of the fundamental particles is exactly related to the dimensional coupling constant by a formula of Al. Zamolodchikov's type

$$M = \frac{4(1-\beta^2)}{\pi\beta^2}\left(\frac{\pi}{2}\mu\gamma\left(\frac{1-\beta^2}{2}\right)\right)^{\frac{1}{1-\beta^2}}\,, \quad (2.4)$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

3. Suzuki equations

In this section we shall use more appropriate parameters for the lattice case:

$$\nu = \tfrac{1}{2}(1-\beta^2) \qquad q = e^{\pi i \nu}\,.$$

Consider an inhomogeneous XXZ chain of spin 1 of even length L with twist q^κ. In order to avoid multiple change of variables we shall work from the very beginning with the rapidity-like ones. The relation to usual multiplicative variables λ [3] is $\lambda = e^{\pi i\nu\theta}$. We shall consider two transfer-matrices corresponding to auxiliary spaces of spins $1/2$ and 1. Corresponding ground state eigenvalues will be denoted respectively by $T_1(\theta)$, $T_2(\theta)$.

The Baxter equations take the form

$$T_1(\theta)Q(\theta) = a(\theta)Q(\theta+\pi i) + d(\theta)Q(\theta-\pi i)\,, \quad (3.1)$$

where $a(\theta)$ and $d(\theta)$ are trigonometric polynomials:

$$a(\theta) = \prod_{j=1}^{L}\sinh\nu(\theta-\tau_j-\pi i), \quad d(\theta) = \prod_{j=1}^{L}\sinh\nu(\theta-\tau_j+\pi i)\,.$$

$T_1(\theta)$ is a trigonometric polynomial of the same form and the same degree, finally

$$Q(\theta) = e^{\nu\kappa\theta}\prod_{j=1}^{m}\sinh\nu(\theta-\sigma_j)\,, \tag{3.2}$$

σ_j being the Bethe roots. We shall be interested in the case of real τ_j and κ which implies

$$\overline{a(\theta)} = d(\bar{\theta})$$

We are interested in the ground state for which $m = L$. For large L and sufficiently small κ the Bethe roots are close to the two-strings: $\sigma_{2j-1} \simeq \eta_j - \pi i/2$, $\sigma_{2j} \simeq \eta_j + \pi i/2$ for certain real η_j.

The transfer-matrix $T_2(\theta)$ is obtained by the fusion relation:

$$T_2(\theta) = T_1(\theta - \pi i/2)T_1(\theta + \pi i/2) - f(\theta)\,, \tag{3.3}$$

here and later

$$f(\theta) = a(\theta - \pi i/2)d(\theta + \pi i/2)\,.$$

According to the investigation done by Suzuki [8] the zeros of $T_1(\theta)$ lie approximately on the lines $\mathrm{Im}(\theta) = \pm 3\pi i/2$, and zeros of $T_2(\theta)$ lie approximately on the lines $\mathrm{Im}(\theta) = \pm\pi i$, $\mathrm{Im}(\theta) = \pm 2\pi i$.

Let us introduce the auxiliary functions

$$y(\theta) = \frac{T_2(\theta)}{f(\theta)}, \quad Y(\theta) = 1 + y(\theta)\,. \tag{3.4}$$

The function $\log(T_2(\theta))$ grows for $\mathrm{Re}(\theta) \to \pm\infty$ slowly (as $\pm 2L\theta$). This allows to derive from (3.3) the first important relation:

$$\log T_1(\theta) = (L * \log(fY))(\theta)\,, \tag{3.5}$$

where we introduced the kernel which will be often used:

$$L(\theta) = \frac{1}{2\pi\cosh\theta}\,,$$

and $*$ means the usual convolution product.

We have

$$T_2(\theta) = \lambda_1(\theta) + \lambda_2(\theta) + \lambda_3(\theta)\,,$$

where

$$\lambda_1(\theta) = a(\theta + \pi i/2)a(\theta - \pi i/2)\frac{Q(\theta + 3\pi i/2)}{Q(\theta - \pi i/2)}$$

$$\lambda_2(\theta) = a(\theta + \pi i/2)d(\theta - \pi i/2)\frac{Q(\theta - 3\pi i/2)Q(\theta + 3\pi i/2)}{Q(\theta - \pi i/2)Q(\theta + \pi i/2)}$$

$$\lambda_3(\theta) = d(\theta + \pi i/2)d(\theta - \pi i/2)\frac{Q(\theta - 3\pi i/2)}{Q(\theta + \pi i/2)}\,.$$

The second auxiliary function is defined by

$$b(\theta) = \frac{\lambda_1(\theta + \pi i/2) + \lambda_2(\theta + \pi i/2)}{\lambda_3(\theta + \pi i/2)}, \quad B(\theta) = 1 + b(\theta)\,. \tag{3.6}$$

Using the Baxter equation we derive

$$b(\theta)=T_1(\theta)\frac{Q(\theta+2\pi i)}{Q(\theta-\pi i)}\frac{a(\theta+\pi i)}{d(\theta)d(\theta+\pi i)}\,. \tag{3.7}$$

On the other hand it is obvious from the definition that

$$T_2(\theta+\pi i/2)=B(\theta)d(\theta+\pi i)d(\theta)\frac{Q(\theta-\pi i)}{Q(\theta+\pi i)}\,. \tag{3.8}$$

Multiplying the latter equation by the conjugated one for real θ one easily derives the second important equation

$$\log y(\theta)=(L*\log(B\overline{B}))(\theta)\,. \tag{3.9}$$

Now comes the main of Suzuki's tricks. Consider a function $G(\theta)$ which is regular in the strip $0<\mathrm{Im}(\theta)<\pi$, and which decrease sufficiently fast at $\pm\infty$. Then having in mind the structure of zeros of $T_2(\theta)$ described above we have

$$\int\limits_{-\infty}^{\infty}\big(G(\theta-\theta')\log T_2(\theta'+\pi i/2)-G(\theta-\theta'+\pi i)\log T_2(\theta'-\pi i/2)\big)\,d\theta'=0\,. \tag{3.10}$$

Using (3.8) we rewrite this as follows

$$\begin{aligned}&\int\limits_{-\infty}^{\infty}\big(G(\theta-\theta')+G(\theta-\theta'+\pi i)\big)\log\frac{Q(\theta'+\pi i)}{Q(\theta'-\pi i)}d\theta'\\&=\int\limits_{-\infty}^{\infty}\big(G(\theta-\theta')\log(d(\theta')d(\theta'+\pi i))-G(\theta-\theta'+\pi i)\log(a(\theta')a(\theta'-\pi i))\big)\,d\theta'\\&+\int\limits_{-\infty}^{\infty}\big(G(\theta-\theta')\log(B(\theta'))-G(\theta-\theta'+\pi i)\log(\overline{B}(\theta'))\big)\,d\theta'\,.\end{aligned}$$

The goal now is to rewrite the left hand side in terms of the auxiliary function $y(\theta),b(\theta)$. From (3.7) and (3.5) one derives

$$\begin{aligned}\log b(\theta)&=\log\left(\frac{Q(\theta+2\pi i)}{Q(\theta-\pi i)}\right)+\log\left(\frac{a(\theta+\pi i-i0)}{d(\theta)d(\theta+\pi i)}\right)\\&\quad+\int\limits_{-\infty}^{\infty}L(\theta-\theta')\log\big(f(\theta')Y(\theta')\big)\,d\theta'\,.\end{aligned}$$

So, our goal will be achieved if we find such $G(\theta)$ that

$$\begin{aligned}\int\limits_{-\infty}^{\infty}\big(G(\theta-\theta')+G(\theta-\theta'+\pi i)\big)\log\frac{Q(\theta'+\pi i)}{Q(\theta'-\pi i)}&=\log\left(\frac{Q(\theta+2\pi i)}{Q(\theta-\pi i)}\right)\\&\quad+\pi i\nu\kappa(4G_0-3)\,,\end{aligned} \tag{3.11}$$

where the last term takes account of the multiplier $e^{\nu\kappa\theta}$ in $Q(\theta)$, G_0 being the average of G over the real line. Recalling that in the formula for $Q(\theta)$ (3.2) the Bethe roots are approximately two-string one easily finds $G(\theta)$ by Fourier transform:

$$G(\theta)=\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\frac{\sinh\left(\frac{\pi k}{2\nu}(1-3\nu)\right)}{\sinh\left(\frac{\pi k}{2\nu}(1-2\nu)\right)\cosh\left(\frac{\pi k}{2}\right)}e^{-ik\theta}dk\,. \tag{3.12}$$

Notice that $G_0=\frac{1-3\nu}{2(1-2\nu)}$.

Finally, after some computation we arrive at

$$\log b(\theta)=2\sum_j\log\Bigl(\tanh\frac{1}{2}(\theta-\tau_j-i0)\Bigr)-\frac{\pi i\nu\kappa}{1-2\nu} \tag{3.13}$$
$$+(L*\log Y)(\theta)+(G*\log B)(\theta)-(G*\log\overline{B})(\theta+\pi i)\,.$$

We obtain the massive relativistic model from the inhomogeneous lattice one by the usual prescription: set $\tau_j=(-1)^j\tau$ and consider the limit

$$\tau\to\infty\,,\quad L\to\infty,\quad 2Le^{-\tau}\to 2\pi MR\ \ \text{finite}\,.$$

In this limit

$$2\sum_j\log\Bigl(\tanh\frac{1}{2}(\theta-\tau_j)\Bigr)\to -2\pi MR\cosh(\theta)\,.$$

The idea is that in this limit we should obtain the eigenvalue of the transfer-matrix corresponding to the NS ground state with the twist defined by

$$\sqrt{2}\beta P=\nu\kappa\,. \tag{3.14}$$

Here $\sqrt{2}$ comes from the normalisation of the topological charge consistent with (2.1). The normalisation of this twist is explained by the requirement that in the high temperature limit $R\to 0$ the eigenvalue of the first integral of motion, I_1, which is nothing but $L_0-c/24$ is given by

$$i_1=P^2-\frac{1}{16}\,.$$

4. Numerical work

The function $b(\theta)$ rapidly decreases when $\mathrm{Re}(\theta)\to\pm\infty$, $0>\mathrm{Im}\theta>-\pi/2$. Introducing the shift $0<\pi\gamma<\pi/2$ and moving the contours of integration we arrive at the system which allows a numerical investigation:

$$\log b(\theta-\pi i\gamma)=-2\pi MR\cosh(\theta-\pi i\gamma)-\frac{\pi i\sqrt{2}}{\beta}P+\tfrac{1}{2}\log 2 \tag{4.1}$$
$$+\int\limits_{-\infty}^{\infty}L(\theta-\theta'-\pi i\gamma)\log\left(\tfrac{1}{2}Y(\theta')\right)d\theta'$$
$$+\int\limits_{-\infty}^{\infty}\Bigl[G(\theta-\theta')\log B(\theta'-\pi i\gamma)-G(\theta-\theta'+\pi i(1-2\gamma))\log\overline{B(\theta'-\pi i\gamma)}\Bigr]d\theta'\,,$$

$$\log y(\theta)=\int\limits_{-\infty}^{\infty}2\mathrm{Re}\Bigl[L(\theta-\theta'+\pi i\gamma)\log B(\theta'-\pi i\gamma)]d\theta'\,. \tag{4.2}$$

The integrals containing $\log B$ converge at infinities very fast because the absolute value of the integrand is estimated as $\exp(-Const \cdot e^{|\theta|})$ with positive $Const$. The integral with $\log(\frac{1}{2}Y)$ converges much more slower because $y(\theta)$ behaves as $1 + O(e^{-|\theta|})$. In the numerical computations we replace integrals by finite sums, and the above estimates mean that the number of points needed for the approximation of the integral containing $\log(\frac{1}{2}Y)$ should be bigger than that for the integrals containing $\log B$.

Our goal is to consider the high temperature limit $R \to 0$. The previous formulae are simplified if we use the parametrisation:

$$R = \frac{\beta}{\sqrt{2}}\left(\frac{\pi}{2}\mu\gamma\left(\frac{1-\beta^2}{2}\right)\right)^{-\frac{1}{1-\beta^2}} e^{-\theta_0}\,, \tag{4.3}$$

with θ_0 being a dimensionless parameter. Now the driving term in the equation (4.1) becomes

$$-4\sqrt{2}\frac{1-\beta^2}{\beta}e^{-\theta_0}\cosh(\theta - i\gamma)\,.$$

The local integrals of motion are extracted form $y(\theta)$ which is the normalised transfer-matrix of auxiliary spin 1 (3.4). Namely, for $\theta \to \infty$ the asymptotical formula holds:

$$\log y(\theta) \simeq \sum_{k=1}^{\infty} C_{2k-1} i_{2k-1}(\theta_0) e^{-(2k-1)\theta}\,, \tag{4.4}$$

similarly the asymptotics for $\theta \to -\infty$ is related to $\bar{i}_{2k-1}(x)$. The constants C_m are given by

$$C_m = -\frac{\beta}{\sqrt{2}(1-\beta^2)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{1-\beta^2}m\right)}{(m-1)!\left(\frac{m+1}{2}\right)!\Gamma\left(1+\frac{\beta^2}{1-\beta^2}m\right)}\,. \tag{4.5}$$

This normalisation is chosen for the sake of the conformal limit, the appearance of this kind of coefficients is not surprising for a reader familiar with [7], we shall give more explanation in the next section.

The main advantage of the above normalisation is that in the high temperature limit we have

$$e^{-(2k-1)\theta_0} i_{2k-1}(\theta_0) \underset{\theta_0\to\infty}{\to} i_{2k-1}\,,$$

with i_{2k-1} being the local integrals of motion for the CFT case normalised as follows:

$$i_{2k-1} = P^{2k} + \cdots\,.$$

Now we start the numerical work. Our goal is to obtain the formulae for i_1, i_3, i_5 by interpolation in P and ν. This may sound as a purely academic exercise having in mind that these formulae can be obtained analytically as explained in the next section. However, in our further study we shall need to guess the formulae for the one-point functions in the integrable basis of supersymmetric CFT, which are unknown. That is why we want to be sure that our numerical methods are sufficiently precise.

The twist P cannot be too large, we restrict ourselves to $P \le 0.2$, we take β sufficiently close to 1. For given β we interpolate in P from the solutions to (4.1), (4.2) for $\theta_0 = 18$. Integrals are replace by sums with step 0.1, the shift is $\gamma = 0.1$, the limits in the integrals containing $\log B(\theta - \pi i\gamma)$ are $[-24, 24]$, the limits of the integral containing $\log(Y(\theta)/2)$ are $[-72, 72]$.

We normalise by the leading coefficient which is later compared with C_{2k-1}. Doing that for a sufficient number of different β's and assuming that due to the general structure of CFT the local integrals must be polynomials in

$$Q^2 = -\frac{(1-\beta^2)^2}{\beta^2}\,,$$

we were able to interpolate further:

$$\begin{aligned} i_1 &= P^2 - \frac{1}{16} \\ i_3 &= P^4 - \frac{5}{16}P^2 + \frac{1}{512}(9+2Q^2)\,, \\ i_5 &= P^6 - \frac{35}{48}P^4 + \frac{537+46Q^2}{3072}P^2 - \frac{475+190Q^2+24Q^4}{49152}\,. \end{aligned} \tag{4.6}$$

We shall not go into the details of the interpolation restricting ourselves to two examples in which we compare the results of the numerical computations using the equations (4.1), (4.2) with the analytical formulae (4.5), (4.6).

It is more direct to compare computational results with

$$j_m = C_m i_m\,.$$

Here are the results for $\beta^2 = \frac{1}{2}$:

P	j_1 comp.	j_3 comp.	j_5 comp.	j_1 analyt.	j_3 analyt.	j_5 analyt.
0.02	0.195092899	−0.121737971	0.385270717	0.195092904	−0.121737972	0.385270720
0.04	0.191322988	−0.118811577	0.375422434	0.191322993	−0.118811578	0.375422438
0.06	0.185039803	−0.113984520	0.359237764	0.185039807	−0.113984521	0.359237767
0.08	0.176243343	−0.107332198	0.337056416	0.176243348	−0.107332199	0.337056419
0.1	0.164933610	−0.0989601675	0.309346006	0.164933614	−0.0989601686	0.309346008
0.12	0.151110603	−0.0890041464	0.276694070	0.151110607	−0.0890041473	0.276694072
0.14	0.134774321	−0.0776300103	0.239797812	0.134774325	−0.0776300111	0.239797814
0.16	0.115924766	−0.0650337947	0.199451558	0.115924769	−0.0650337954	0.199451559
0.18	0.0945619364	−0.0514416943	0.156531934	0.0945619389	−0.0514416947	0.156531935
0.2	0.0706858328	−0.0371100629	0.111980775	0.0706858347	−0.0371100632	0.111980775

Here are the results for $\beta^2 = \frac{3}{5}$:

P	j_1 comp.	j_3 comp.	j_5 comp.	j_1 analyt.	j_3 analyt.	j_5 analyt.
0.02	0.267141860	−0.315491660	1.87869822	0.267141961	−0.315491728	1.87869854
0.04	0.261979700	−0.308328920	1.83430033	0.261979797	−0.308328984	1.83430063
0.06	0.253376100	−0.296514050	1.76131551	0.253376191	−0.296514109	1.76131579
0.08	0.241331061	−0.280231598	1.66124365	0.241331143	−0.280231651	1.66124390
0.1	0.225844581	−0.259739931	1.53614935	0.225844653	−0.259739975	1.53614955
0.12	0.206916661	−0.235371233	1.38862669	0.206916720	−0.235371267	1.38862685
0.14	0.184547302	−0.207531507	1.22175395	0.184547345	−0.207531531	1.22175405
0.16	0.158736503	−0.176700578	1.03903821	0.158736527	−0.176700590	1.03903826
0.18	0.129484265	−0.143432085	0.844349942	0.129484268	−0.143432087	0.844349948
0.2	0.0967905868	−0.108353490	0.641847507	0.0967905654	−0.108353481	0.641847468

It is clear from these tables that the agreement is quite good. It can be made better by choosing bigger θ_0, using finer discretisation *etc.* But this is not needed for our goals since our precision was sufficient for a successful interpolation.

5. Eigenvalues of integrals from ODE-CFT correspondence

The ODE-CFT correspondence is the statement that in the conformal case the vacuum eigenvalues of the operator $Q(\theta)$ coincide with determinants of certain ordinary differential equations. The eigenvalues of the transfer-matrices $T_j(\theta)$ coincide with certain Stokes multipliers for the corresponding equation. In the case of $c < 1$ CFT this statement goes back to a remarkable observation due to Dorey ans Tateo [10], which was later essentially clarified and generalised by Bazhanov, Lukyanov, Zamolodchikov [11]. We shall not go into details of further generalisation of the ODE-CFT correspondence and its generalisation to the massive case, restricting ourselves to the case of supersymmetric CFT which is considered in the present paper. It is useful to consider more general situation of a parafermion Ψ_k interacting with a free boson because there is certain difference between k even or odd. The $c = 1$ CFT corresponds to $k = 1$, and the $c = 3/2$ case, considered in this paper, corresponds to $k = 2$. In general case Lukyanov [12] proved that the operator $Q(\theta)$ is related to the following ODE:

$$\psi''(z) - \Big((z^{2\alpha} - E)^k + \frac{l(l+1)}{z^2}\Big)\psi(z) = 0\,, \tag{5.1}$$

the relation of E, α, l to parameters θ, β^2, k, P is as follows

$$\alpha = \frac{1-\beta^2}{k\beta^2}\,, \quad E = \frac{\beta}{\sqrt{k}} e^{\frac{1-\beta^2}{k}(\theta-\theta_0)}\,, \quad l = \frac{\sqrt{k}}{\beta}P - \frac{1}{2}\,, \tag{5.2}$$

and θ_0 is defined by a formula analogous to (4.3). The parameter α is positive, so, we are dealing with a self-adjoint operator on the positive half-line. Then $Q(E)$ is just its determinant (here and later we allow ourselves to use both $Q(\theta)$ and $Q(E)$ having in mind the identification (5.2)).

The eigenvalues $Q(E)$ and $T_j(E)$ are entire functions of E. We are interested in their large E asymptotics. It is known that for $\log Q(E)$ and for $\log T_j(E)$ with j up to $k-1$ the asymptotics go in two kinds of exponents: $E^{-\frac{2j-1}{2k(1-\beta^2)}}$ and $E^{\frac{j}{k\beta^2}}$, $(j \geq 1)$, the coefficients being proportional to the eigenvalues of local and non-local integrals of motion. The latter are of no interest for us, that is why we shall deal directly with $\log T_k(E)$ which possesses an exceptional property of containing in its asymptotics $E^{-\frac{2j-1}{2k(1-\beta^2)}}$ only. In order to explain that we have to consider (5.1) as an equation of a complex variable.

Let $z = |z|e^{i\varphi}$. Since the parameter α is generally irrational we are dealing with an infinite covering of the plane: $-\infty < \varphi < \infty$.

The main property allowing to investigate the determinant and the Stokes multipliers is that for any solution $\psi(z, E)$ the function

$$(\Omega\psi)(z, E) = q^{1/2}\psi(pz, q^2E)\,; \quad p = e^{\pi i\beta^2}\,, \quad q = e^{\pi i\frac{1-\beta^2}{k}}\,,$$

is also a solution.

Consider the solution $\chi(z, E)$ characterised by the following asymptotics for real $z \to +\infty$:

$$\chi(z, E) \simeq x^{-\frac{\alpha k}{2}} \exp\Big(-\frac{x^{\alpha k+1}}{\alpha k + 1}\Big)\,.$$

Following the [11,12] and using the fusion relations it is not hard to derive for any j the relation between the three solutions:

$$(\Omega^{j+1}\chi)(z, E) = -T_{j-1}(Eq^{j+1})\chi(z, E) + T_j(Eq^j)(\Omega\chi)(z, E)\,.$$

The asymptotic behaviour at $E \to \infty$ is investigated by WKB method, where the important role is played by the function $\sqrt{(x^\alpha - E)^k + \frac{l(l+1)}{x^2}}$.

One rescales x for large E so that the term $\frac{l(l+1)}{x^2}$ is small. It is clear that exactly for $j = k$ the function $T_k(Eq^k)$ can be considered as the Stokes multiplier between growing solutions $(\Omega\chi)(z, E)$ and $(\Omega^{k+1}\chi)(z, E)$ for two neighbouring sectors which are semi-classically separated by the cut of the square root. This implies a simple formula for the asymptotics of $\log T_k(Eq^k)$ given below.

Let us change variables rewriting (5.1) as

$$a^2\psi''(x) - \Big((x^{2\alpha} - 1)^k + a^2\frac{l(l+1)}{x^2}\Big)\psi(x) = 0\,, \tag{5.3}$$

where $a^2 = E^{-\frac{k}{(1-\beta^2)}}$.

We prefer to write the WKB formulae in a somewhat XIX century way in order to avoid some total derivatives. Namely, we present the solution to (5.3) in the form

$$\psi(x, x_0) = S(x, a)^{\frac{1}{2}} \exp\Big(\frac{1}{a}\int\limits_{x_0}^{x} \frac{dy}{S(y, a)}\Big),$$

where $S(x, a)$ satisfies the Riccati equation (we omit arguments)

$$\frac{4}{a^2}\Big(1 - FS^2\Big) - S'^2 + 2S''S + x^{-2}S^2 = 0\,,$$

with

$$F(x, a, b) = (x^{2\alpha} - 1)^k + \frac{b^2}{x^2}\,,$$

where we introduce $b = a(l + 1/2)$, in spite of the fact that $b \ll 1$ it is convenient to develop into series in this parameter only at the final stage. The ansatz for ψ is different from usual quantum mechanical formulae, and it allows to avoid appearance of redundant total derivatives. Using Riccati equation we find for $S(x, a)$ the power series

$$S(x, a, b) = \sum_{k=0}^{\infty} a^{2k} S_k(x, b)\,. \tag{5.4}$$

In particular,

$$\frac{1}{S_0(x, b)} = \sqrt{F(x, a, b)}\,.$$

According to our reasoning concerning the Stokes multiplier, we have for the asymptotics

$$\log T_k(Eq^k) \simeq \frac{1}{a}\int\limits_C \frac{dy}{S(y, a)}\,, \tag{5.5}$$

where the contour C goes from $\infty \cdot e^{+i0}$ to $\infty \cdot e^{-i0}$ around the cut of $\sqrt{F(x, a, b)}$. Let us consider the contribution from $S_0(x, b)$. Recalling that $b \ll 1$ we develop

$$\frac{1}{S_0(x, b)} = \sum_{p=0}^{\infty} \binom{1/2}{p} (x^{2\alpha} - 1)^{\frac{k(1-2p)}{2}} b^{2p} x^{-2p}\,.$$

Now the difference between k odd or even becomes clear. We have to evaluate the integral

$$\int_C (y^{2\alpha}-1)^{\frac{k(1-2p)}{2}} y^{-2p} dy .$$

By the change of variables $w = y^{2\alpha}$ this integral reduces for odd k to a beta-function and for even k to a binomial coefficient. In spite of this computational difference the final result does not depend on the parity of k, after some simplification we get

$$\int_C (y^{2\alpha}-1)^{\frac{k(1-2p)}{2}} y^{-2p} dy = \frac{\pi i k\beta^2}{1-\beta^2} e^{-\frac{\pi i}{2}k(2p-1)} \frac{\Gamma\left(\frac{k(2p-1)}{2(1-\beta^2)}\right)}{\Gamma\left(1+\frac{k\beta^2(2p-1)}{2(1-\beta^2)}\right)\Gamma\left(\frac{k(2p-1)}{2}\right)} .$$

Plugging this into (5.5) we find the constants C_m. Higher corrections in a^2 following from (5.4) are considered similarly. For $k=2$ one finds exactly the expressions (4.6).

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