Anisotropic models and angular moments methods for the Compton scattering

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C. Buet[*], B. Després[†] and T. Leroy[‡]

Abstract

This paper is devoted to the derivation of models for the Compton scattering. These models generalize the Kompaneets equation [KOM57] to anisotropic distributions. The P_1 angular method is applied, leading to a system of nonlinear Fokker-Planck equations. We show that there is a limit regime in which this P_1 model exhibits new solutions, in comparison with the Bose condensate result of Caflish and Levermore [CL86] for the Kompaneets equation.

1 Introduction

The general aim of this paper is a hierarchy of kinetic models describing the Compton scattering. On the one hand, several kinetic models have been derived from the Boltzmann equation by physicists [BPR69, BPR70, PL97, POM73, FKM85] for the so-called imperfect Lorentz gas [BDD56], leading to several anisotropic Fokker-Planck type equations. On the other hand, Escobedo et al [EMV03] recently derived from the Boltzmann equation for photons the Kompaneets equation [KOM57, EMV98], which is a nonlinear Fokker-Planck type equation, by assuming the isotropy of the distribution function and by using several original technics. In this paper we extend this approach to anisotropic distribution functions. This leads to Fokker-Planck equations, whose structure can be seen as anisotropic Kompaneets type equations and to an original P_1 approximation with two coupled Kompaneets type equations. On a complexity scale, the hierarchy of models is described in Table 1.

Boltzmann model $>$ Pomraning model $>$ Moment model $>$ Kompaneets equation

Table 1: Complexity scale of the hierarchy of models.

Boltzmann equation with Compton scattering. The Compton scattering describes the change of energy and direction of a photon, of momentum $\mathbf{p}_\gamma$ interacting with an electron, of momentum $\mathbf{p}_e$, leading to another photon and another electron of momentum $\mathbf{p}'_\gamma$ and $\mathbf{p}'_e$ respectively. The Compton scattering between photons and electrons can be described by a Boltzmann equation for the density distribution function of the photons. Assuming that the electrons are at thermodynamic equilibrium, their distribution function is given by a Maxwellian with temperature $T > 0$. Since the induce effects (quantum effects) are taken into account for the photons, the collision operator is quadratic with respect to the distribution function. The density distribution

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of the photons f depends on the time t, on the frequency of the photon $\nu \in \mathbb{R}^+$, on its space position $\mathbf{x} \in \mathbb{R}^3$ and on its direction $\mathbf{\Omega} \in S^2$

$$\frac{1}{c}\partial_t f + \mathbf{\Omega}.\nabla f = \nu^{-2} Q(f). \tag{1.1}$$

As in [BPR70, EMV03] the collision operator is

$$Q(f) = \sigma_s \int_{\mathbb{R}^+ \times S^2} \nu' \nu \big(1 + \cos^2 \theta\big) |\omega|^{-1} e^{-\frac{A^2 mc^2}{2kT|\omega|^2}} e^{\frac{h\nu'}{kT}} q(f) d\nu' d\mathbf{\Omega}' \tag{1.2}$$

with

$$\begin{cases} A = \dfrac{h\nu'}{c} - \dfrac{h\nu}{c} + \dfrac{|\omega|^2}{2mc}, \\ \omega = \dfrac{h\nu'}{c}\mathbf{\Omega}' - \dfrac{h\nu}{c}\mathbf{\Omega}, \\ \cos\theta = \mathbf{\Omega}.\mathbf{\Omega}', \end{cases} \tag{1.3}$$

and

$$q(f) = e^{-\frac{h\nu}{kT}} f(\nu', \mathbf{\Omega}')\big(1 + f(\nu, \mathbf{\Omega})\big) - e^{-\frac{h\nu'}{kT}} f(\nu, \mathbf{\Omega})\big(1 + f(\nu', \mathbf{\Omega}')\big), \tag{1.4}$$

where the space dependence has been removed for ease of notations. The mass of electrons is $m > 0$. The Boltzmann constant is $k > 0$ and the Planck constant is $h > 0$. The parameter σ_s is a scattering coefficient and is assumed to depend only on the space variable $\mathbf{x}$. Equations (1.3) come from classical collisional identities [BPR69, BPR70] for the collision of a photon and an electron: the parameter ω (resp. θ) represents the transfer of impulsion (resp. variation of angle) between the incoming and outgoing photons; the global conservation of impulsion and energy is expressed through the following relations $\frac{h\nu}{c}\mathbf{\Omega} + \mathbf{p}_e = \frac{h\nu'}{c}\mathbf{\Omega}' + \mathbf{p}'_e$ and $h\nu + \frac{|\mathbf{p}_e|^2}{2m} = h\nu' + \frac{|\mathbf{p}'_e|^2}{2m}$ in which $\mathbf{p}_e$ (resp. $\mathbf{p}'_e$) is the impulsion of the electron before (resp. after) collision. Since the seminal works of J. A. Kompaneets [KOM57], the complexity of the Boltzmann equation (1.1) has motivated, by various means, the design of reduced models. In our work we distinguish three reduced models which are the Kompaneets equation (1.5), the Pomraning Boltzmann equation with simplified collision kernel (1.7) and the original anisotropic moment model (1.8) which we will derive and study.

The Kompaneets equation is at the other end on the complexity scale. It is a Fokker-Planck type equation widely studied in the literature [COO71, POM73, EMV98, EMV04, KAV02]

$$\partial_t f + \mathbf{\Omega}.\nabla f = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu f + f(1+f)\Big)\right]. \tag{1.5}$$

This equation has a certain number of desirable properties inherited from (1.1), we refer to [EMV98] for a complete study. It preserves the non negativity of the distribution function, satisfies a H-theorem and preserves the total number of photons $N(f) = \int \nu^2 f d\nu$ under flux conditions at $\nu = 0$ and $\nu = +\infty$. Moreover the the Bose-Einstein stationary solutions of the space-homogeneous Boltzmann equation (1.1)

$$f_\mu(\nu) = \Big(e^{(\mu+\nu)/T} - 1\Big)^{-1}, \qquad \mu > 0, \tag{1.6}$$

are also stationary solutions of the space-homogeneous Kompaneets equation. Note that $\nu^3 f_0(\nu) = B(\nu)$ is the Planck function: it is the equilibrium state of the transfer equation under the assumption of local thermodynamic equilibrium (ETL) [MWM99]. Theoretical results [CL86, EMV98] from the literature about existence of a solution, long time behavior and the zero flux for the Kompaneets equation condition are reproduced in the Appendix A for the convenience of the reader.

The Pomraning model is an anisotropic equation [POM73] with a simplified collision kernel which reproduces some features of an imperfect Lorentz gas [BDD56]

$$\frac{1}{c}\partial_t f + \mathbf{\Omega}.\nabla f = \frac{\sigma_s}{4\pi} \int_{S^2} \big(1 + \cos^2\theta\big)\big(f' - f\big) d\mathbf{\Omega}' + \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu f + f\big(1+f\big)\Big)\right]. \tag{1.7}$$

This model interpolates between the full anisotropic Boltzmann equation (1.1) and the isotropic Kompaneets equation and so is a simplified model of anisotropic effects. It has natural good properties which are: a H-theorem, the conservation of the Bose-Einstein stationary states (1.6), the conservation of the total number of photons and the conservation of the non negativity of the distribution function.

An interesting moment model which approximates the Pomraning equation will finally be considered. It is derived as a P_1 approximation [BRU00, BRU02, HLM10] of (1.7)

$$\left\{\begin{array}{llll} \partial_t E & +\nabla\cdot\mathbf{F} & = \frac{\sigma_s}{3mc^2}\nu^{-2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T\partial_\nu E+E+E^2+3(\mathbf{F},\mathbf{F})\right)\right], \\ \partial_t\mathbf{F} & +\frac{1}{3}\nabla E & = \frac{\sigma_s}{3mc^2}\nu^{-2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T\partial_\nu\mathbf{F}+\mathbf{F}+2E\mathbf{F}\right)\right]-\alpha\mathbf{F}, \end{array}\right. \tag{1.8}$$

where E (resp. $\mathbf{F}$) is the zero-th (resp. first) order moment of the distribution function f and $\alpha \geq 0$ is a phenomenological parameter. This phenomenological parameter yields isotropization since it damps only $\mathbf{F}$ and it is called the friction parameter in this work. The zero-th order moment satisfies a Kompaneets equation, perturbed by a Burgers type term on the first order moment. The evolution of the first order moment $\mathbf{F}$ involves a competition between a Fokker-Planck and the friction term. For $\alpha = 0$, we show that the threshold value on the initial photons' number for which a condensation phenomena appear (refer to Caflish-Levermore [CL86]) is different to the one of the Kompaneets equation. A major asset of this new model is that it can be used to explore the dynamics of anisotropic Kompaneets flows at reduced numerical cost.

Our main results are threefold. Firstly in Theorem 8, we prove with a simple scaling with respect to the small parameter $\varepsilon = \frac{kT}{mc^2}$ that the full Boltzmann model (1.1-1.3) tends to the Kompaneets equation (1.5) in the sense of distribution for isotropic and homogeneous profiles. Contrary to [EM01, EMV03, EMV98] our proof does not need any modeling of some coefficients, and in this sense, it is simpler and more rigorous. Secondly, for $\alpha = 0$, we show how to diagonalize the anisotropic moment model (1.8) in the form of two decoupled Kompaneets equations. Relying on earlier results of [CL86, EM01, EMV03, EMV98], it gives access to the long-time dynamics of this model in Theorem 14. Thirdly we discretize the moment model (1.8) and illustrate the dynamics of of anisotropic Kompaneets flows for different values of the phenomenological parameter ($\alpha = 1$, 0.1 and 0).

Organization of this work Theorem 8 is proved in Section 2. The Pomraning model (1.7) is studied in Section 3. This equation is proved to satisfy a H-theorem and the conservation of the non negativity of the distribution function. Section 4 is devoted to the design and study of the P_1 model (1.8). We prove in Theorem 14 that in the limit case $\alpha = 0$, the P_1 model may exhibit new solutions with respect to the Caflish-Levermore long time regime [CL86]. Numerical illustrations with this new model are proposed at the end of the Section.

Useful simplifications For the ease of notations, we focus on space-homogeneous profiles, drop out the derivatives in space and focus on the major difficulty which is the collision operator (1.2-1.3). Additionally we will use the non dimensional formulation introduced in the next section.

2 From Boltzmann to Kompaneets

In this section we study the convergence of the Botzmann equation (1.1) to the Kompaneets equation (1.5) in the sense of distributions and prove the Theorem 8 after convenient rescaling of the equations. We start from the Boltzmann equation (1.1). The so-called detailed balance law writes

$$b(x,\nu,\nu',\theta)e^{h\nu'/kT} = b(x,\nu',\nu,\theta)e^{h\nu/kT} = |\omega|^{-1}e^{-\frac{mh^2}{2kT|\omega|^2}|\nu'-\nu|^2-\frac{|\omega|^2}{8mkT}+\frac{h\nu'+h\nu}{2kT}}$$

We introduce a parameter $\varepsilon = kT/mc^2 << 1$ and renormalize the variables and opacity

$$\bar{\nu} = \frac{h\nu}{kT}, \quad \bar{\nu}' = \frac{h\nu'}{kT}, \quad \bar{t} = \frac{ct}{L}, \quad \bar{x} = \frac{x}{L} \quad \text{and } \bar{\sigma}_s = \frac{cL\varepsilon^{\frac{3}{2}}}{h}\sigma_s.$$

Set $\bar{f}(t, \bar{\mathbf{x}}, \bar{\nu}, \mathbf{\Omega}) = f(t, \mathbf{x}, \nu, \mathbf{\Omega})$ and rescale $\bar{\omega} = \frac{c}{kT}\omega = \bar{\nu}'\mathbf{\Omega}' - \bar{\nu}\mathbf{\Omega}$. Discarding the space dependance, introduce

$$\bar{B}_\varepsilon(\bar{\nu}, \bar{\nu}', \theta) = \frac{kT}{c} b(\nu, \nu', \theta) e^{\nu'} = |\bar{\omega}|^{-1} e^{-\frac{1}{2\varepsilon}\frac{|\bar{\nu}'-\bar{\nu}|^2}{|\bar{\nu}'\mathbf{\Omega}'-\bar{\nu}\mathbf{\Omega}|^2} - \frac{\varepsilon}{8}|\bar{\nu}'\mathbf{\Omega}'-\bar{\nu}\mathbf{\Omega}|^2 + \frac{\bar{\nu}+\bar{\nu}'}{2}}. \tag{2.1}$$

It yields the equation $\partial_t \bar{f} = \frac{1}{\bar{\nu}^2}\bar{Q}_\varepsilon(\bar{f})$ with the collision operator $\bar{Q}_\varepsilon(\bar{f}) = \bar{\sigma}_s \int_{\mathbb{R}^+\times S^2} \bar{\nu}'\bar{\nu}\big(1 + \cos^2\theta\big)\bar{B}_\varepsilon(\bar{\nu}, \bar{\nu}', \theta)\bar{q}(\bar{f})d\bar{\nu}'d\mathbf{\Omega}'$. Dropping the bars for ease of notations, one obtains the **non dimensional space-homogeneous Boltzmann equation with a small parameter** ε

$$\nu^2 \frac{d}{dt} f = Q_\varepsilon(f) := \sigma_s \frac{1}{\varepsilon^{\frac{3}{2}}} \int_{\mathbb{R}^+\times S^2} \nu'\nu\big(1 + \cos^2\theta\big)B_\varepsilon(\nu, \nu', \theta)q(f)d\nu'd\mathbf{\Omega}'. \tag{2.2}$$

2.1 Preliminary considerations

We aim to study the limit as $\varepsilon \to 0$ of the right hand side, and for the sake of simplicty we set $\sigma_s = 1$. We notice that the weight $|\omega|^{-1}$ is singular in B_ε. Nevertheless one has

$$\begin{aligned} |\omega|^2 &= \nu^2 + \nu'^2 - 2\nu\nu'\cos\theta, \\ &= (\nu - \nu')^2 + 2\nu\nu'(1 - \cos\theta), \\ &= (\nu - \nu')^2 + 4\nu\nu'\sin^2\theta/2, \end{aligned} \tag{2.3}$$

We refer to [EMV03] for a study of the limit of this collision operator as ε tends to zero. However we found that an alternative is to pass to the limit in the weak sense. We also believe it is simpler and ultimately more rigorous. Therefore we consider a test function and integrate the Boltzmann operator in (2.2). It defines

$$H_\varepsilon(\psi) = \int_{\mathbb{R}^+\times S^2} Q_\varepsilon(f)\psi(\nu, \mathbf{\Omega})d\nu d\mathbf{\Omega}$$

$$= \frac{1}{\varepsilon^{\frac{3}{2}}} \int_{\mathbb{R}^+\times S^2}\int_{\mathbb{R}^+\times S^2} \nu'\nu\big(1 + \cos^2\theta\big)B_\varepsilon(\nu, \nu', \theta)q(f)\psi(\nu, \mathbf{\Omega})d\nu d\nu' d\mathbf{\Omega}' d\mathbf{\Omega}.$$

The quantity q defined in (1.4) is a antisymmetric difference $q(f) = a(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') - a(\nu', \nu, \mathbf{\Omega}', \mathbf{\Omega})$, with $a(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') = e^{-\nu} f(\nu', \mathbf{\Omega}')\big(1 + f(\nu, \mathbf{\Omega})\big)$. Let us introduce some notations

$$\begin{cases} b(\nu, \mathbf{\Omega}) = e^{\nu} f(\nu, \mathbf{\Omega}) / \left(1 + f(\nu, \mathbf{\Omega})\right) \\ [b]\,(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') = b(\nu, \mathbf{\Omega}) - b(\nu', \mathbf{\Omega}'), \\ [a]\,(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') = a(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') - a(\nu', \nu, \mathbf{\Omega}', \mathbf{\Omega}), \\ [\psi]\,(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') = \psi(\nu, \mathbf{\Omega}) - \psi(\nu', \mathbf{\Omega}'). \end{cases} \tag{2.4}$$

Note the identity

$$[a] = -(1 + f')(1 + f)e^{-\nu-\nu'}[b]. \tag{2.5}$$

Lemma 1. *The integral $H_\varepsilon(\psi)$ admits the formal reformulation*

$$H_\varepsilon(\psi) = \frac{1}{2\varepsilon^{\frac{3}{2}}} \int_{\mathbb{R}^+\times S^2}\int_{\mathbb{R}^+\times S^2} \nu'\nu\big(1 + \cos^2\theta\big)B_\varepsilon(\nu, \nu', \theta)[a][\psi]d\nu d\nu' d\mathbf{\Omega}' d\mathbf{\Omega}. \tag{2.6}$$

For $e^\nu f \in L^\infty(\mathbb{R}^+ \times S^2)$ and $\psi \in L^\infty(\mathbb{R}^+ \times S^2)$, the term below the integral is integrable.

Proof. From (2.1)-(2.3)-(2.5), there exists a constant $C > 0$ such that for all $0 \le \theta \le \pi$

$$\left|\nu'\nu\big(1 + \cos^2\theta\big)B_\varepsilon(\nu, \nu', \theta)[a][\psi]\right| \le C \frac{\nu\nu' e^{-(\nu+\nu')/2}}{\sqrt{(\nu - \nu')^2 + 2\nu\nu'(1 - \cos\theta)}}.$$

Set

$$R=\int_{\mathbb{R}^+\times\mathbb{R}^+}\left(\int_{S^2\times S^2}\frac{1}{\sqrt{(\nu-\nu')^2+2\nu\nu'(1-\cos\theta)}}d\mathbf{\Omega}d\mathbf{\Omega'}\right)\nu\nu' e^{-(\nu+\nu')/2}d\nu d\nu'$$

Since for any integrable real function g

$$\int_{S^2\times S^2} g(\mathbf{\Omega}.\mathbf{\Omega'})d\mathbf{\Omega}d\mathbf{\Omega'}=\int_{S^2\times S^2} g(\cos\theta)d\mathbf{\Omega}d\mathbf{\Omega'}=8\pi^2\int_0^\pi g(\cos\theta)\sin\theta d\theta \tag{2.7}$$

where $0<\theta<\pi$, one gets

$$R=8\pi^2\int_{\mathbb{R}^+\times\mathbb{R}^+}\left(\int_0^\pi\frac{\sin\theta}{\sqrt{(\nu-\nu')^2+2\nu\nu'(1-\cos\theta)}}d\theta\right)\nu\nu' e^{-(\nu+\nu')/2}d\nu d\nu'.$$

A direct calculation gives

$$\int_0^\pi\frac{\sin\theta d\theta}{\sqrt{(\nu-\nu')^2+2\nu\nu'(1-\cos\theta)}}=\int_{-1}^1\frac{d\mu}{\sqrt{(\nu-\nu')^2+2\nu\nu'(1-\mu)}}=\frac{\min(\nu,\nu')}{\nu\nu'}.$$

So

$$R\le 8\pi^2\int_{\mathbb{R}^+\times\mathbb{R}^+}\min(\nu,\nu')e^{-(\nu+\nu')/2}d\nu d\nu'<\infty.$$

The proof is ended. □

We recall now important properties of the Boltzmann equation (2.2). Firstly, the Boltzmann equation preserves the total number of photons $N(f)=\int_{S^2\times\mathbb{R}^+} f d\mathbf{\Omega}\nu^2 d\nu$, whose proof directly comes from the weak formulation (2.6) of the Boltzmann operator by taking $\psi=1$. Secondly we turn to another important property of the Boltzmann equation, which is the H-theorem: it expresses the non reversibility of the physical process [BC03, CRS08].

Theorem 2 (H-Theorem)**.** *Consider the functional*

$$\mathrm{H(t)}=\int_{S^2\times\mathbb{R}^+}\Big(f\log f-(f+1)\log(f+1)+\nu f\Big)d\mathbf{\Omega}\nu^2 d\nu. \tag{2.8}$$

Assume the solution f of (2.2) *is non negative and is such that f and $\phi(f)$ are integrable for the measure $\nu^2 d\nu d\mathbf{\Omega}$. Then the functional H is monotone non increasing, i.e.*

$$\mathrm{H}'(t)\le 0$$

Bose-Einstein distributions

$$B_\mu(\nu)=\Big(e^{(\mu+\nu)}-1\Big)^{-1}\quad \mu\ge 0, \tag{2.9}$$

are such that

$$\mathrm{H}'(t)=0.$$

Proof. Consider the function $\phi(f)=f\log(f)-(f+1)\log(f+1)+\frac{\nu}{T}f$. One has

$$\phi'(f)=\log\left(\frac{e^\nu f}{1+f}\right)$$

Thus $\mathrm{H}'(t)=\int_{\mathbb{R}^+\times S^2}Q_\varepsilon(f)\nu^2 d\nu d\mathbf{\Omega}$. Using (2.6) with $\psi=\log\left(\frac{e^\nu f}{1+f}\right)$, the third relation of (2.4) and (2.5), one obtains

$$\mathrm{H}'(t)=\frac{-1}{2\varepsilon^{\frac{3}{2}}}\int_{\mathbb{R}^+\times S^2}\int_{\mathbb{R}^+\times S^2}\nu'\nu\big(1+\cos^2\theta\big)B_\varepsilon(\nu,\nu',\theta)(1+f')(1+f)e^{-\nu-\nu'}[b][\log(b)]d\nu d\nu' d\mathbf{\Omega'}d\mathbf{\Omega}.$$

Since the log is a strictly increasing function, that is for any real number a and b , $(a-b)(\log a - \log b \geq 0$ with equality if and only if $a = b$, thus $\mathrm{H}'(t) \leq 0$. For Bose-Einstein distributions, $[b] = 0$ thus $\mathrm{H}'(t) = 0$, so the inequality is an equality. The proof is ended $\square$

The analysis pursues by studying the limit of $H_\varepsilon(\psi)$ as ε tends to 0, taking care of the singularity which comes from $|\omega|$. Introduce the change of variables

$$y = \frac{\nu + \nu'}{\sqrt{2}} \in \mathbb{R}^+, \qquad z = \frac{\nu - \nu'}{\sqrt{2}} \in [-y, y]. \tag{2.10}$$

The inverse transformation is

$$\nu = \frac{y+z}{\sqrt{2}}, \qquad \nu' = \frac{y-z}{\sqrt{2}}. \tag{2.11}$$

With these definition one has the formulas

$$\nu\nu' = \frac{y^2 - z^2}{2},$$
$$|\omega|^2 = |\nu'\mathbf{\Omega}' - \nu\mathbf{\Omega}|^2 = \nu^2 + \nu^{'2} - 2\nu\nu'\cos\theta = 2\big(z^2 + (y^2 - z^2)\sin^2\theta/2\big),$$
$$|\omega|^{-1} e^{-\frac{1}{2\varepsilon}\frac{|\nu'-\nu|^2}{|\nu'\mathbf{\Omega}'-\nu\mathbf{\Omega}|^2} - \frac{\varepsilon}{8}|\nu'\mathbf{\Omega}'-\nu\mathbf{\Omega}|^2 + \frac{\nu+\nu'}{2}} = \frac{e^{-\frac{1}{2\varepsilon}\frac{z^2}{z^2+(y^2-z^2)\sin^2\theta/2}} e^{-\frac{\varepsilon}{4}\left(z^2+(y^2-z^2)\sin^2\theta/2\right)} e^{y/\sqrt{2}}}{\sqrt{2\big(z^2+(y^2-z^2)\sin^2\theta/2\big)}}.$$

The operator (2.6) becomes

$$\begin{aligned} H_\varepsilon(\psi) = \frac{1}{2\varepsilon^{\frac{3}{2}}} \int_{S^2\times S^2} \int_{y=0}^{\infty} \int_{-y}^{y} [a][\psi] \frac{y^2 - z^2}{2} \big(1 + \cos^2\theta\big) \\ \times \frac{e^{-\frac{1}{2\varepsilon}\frac{z^2}{z^2+(y^2-z^2)\sin^2\theta/2}} e^{-\frac{\varepsilon}{4}\left(z^2+(y^2-z^2)\sin^2\theta/2\right)} e^{y/\sqrt{2}}}{\sqrt{2\big(z^2+(y^2-z^2)\sin^2\theta/2\big)}} dydzd\mathbf{\Omega}'d\mathbf{\Omega}. \end{aligned} \tag{2.12}$$

The most important term in this integral is $e^{-\frac{1}{2\varepsilon}\frac{z^2}{z^2+(y^2-z^2)\sin^2\theta/2}}$ which shows strong exponential damping for small $\varepsilon > 0$, except for $z = 0$. At the level of principles, damping can be decomposed between angular damping and frequency damping. Our first goal is to characterize the frequency damping with a measure concentrated on $z = 0 \Leftrightarrow \nu = \nu'$. This is possible as shown with the additional change of variables below.

For fixed y, $\mathbf{\Omega}$ and $\mathbf{\Omega}'$, we define s as a function of z by

$$s = \frac{1}{\sqrt{\varepsilon}} \frac{z}{|\nu'\mathbf{\Omega}' - \nu\mathbf{\Omega}|} = \frac{1}{\sqrt{2\varepsilon}} \frac{z}{\sqrt{z^2 + (y^2 - z^2)\sin^2\theta/2}}. \tag{2.13}$$

With this variable, the term stressed above becomes a Gaussian factor $e^{-\frac{1}{2\varepsilon}\frac{z^2}{z^2+(y^2-z^2)\sin^2\theta/2}} = e^{-s^2}$. The differential relation between s and z is

$$ds = \frac{y^2\sin^2\theta/2}{\sqrt{2\varepsilon}\left(z^2 + (y^2 - z^2)\sin^2\theta/2\right)^{3/2}} dz. \tag{2.14}$$

Since $\dfrac{ds}{dz} \geq 0$, the new variable s is in the interval $s \in [s(-y), s(y)] = \left[-\frac{1}{\sqrt{2\varepsilon}}, \frac{1}{\sqrt{2\varepsilon}}\right]$. Algebraic manipulations from (2.13) shows the inverse transform is correctly defined for $\theta \neq 0$

$$z = \frac{|\sin\theta/2|}{\sqrt{1 - 2\varepsilon s^2(1 - \sin^2\theta/2)}} \sqrt{2\varepsilon} y s. \tag{2.15}$$

Since $|\sin\theta/2| \leq \frac{|\sin\theta/2|}{\sqrt{1-2\varepsilon s^2(1-\sin^2\theta/2)}} \leq 1$ because it can be viewed as an increasing function of s, we note the bound

$$|z| \leq \sqrt{2\varepsilon} y s. \tag{2.16}$$

Next formula has immediate interest in the study of $H_\varepsilon(\psi)$.

Lemma 3. *One has the differential formula*

$$\frac{dz}{\sqrt{z^2(1-\sin^2\theta/2)+y^2\sin^2\theta/2}} = \frac{\sqrt{2\varepsilon}}{1-2\varepsilon s^2(1-\sin^2\theta/2)}ds. \tag{2.17}$$

Proof. The identity (2.13) can be recast as

$$z^2(1-\sin^2\theta/2)+y^2\sin^2\theta/2 = \frac{z^2}{s^2 2\varepsilon} = \frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}. \tag{2.18}$$

So $(1-\sin^2\theta/2)+y^2\sin^2\theta/2\frac{1}{z^2} = \frac{1}{s^2 2\varepsilon}$. Differentiation on both sides yields

$$y^2\sin^2\theta/2\frac{dz}{z^3} = \frac{ds}{s^3 2\varepsilon} \Longleftrightarrow \left(\frac{1}{z/(s\sqrt{2\varepsilon})}\right)dz = \left(\frac{1}{y^2\sin^2\theta/2}\times\frac{z^2}{s^2 2\varepsilon}\right)\sqrt{2\varepsilon}ds.$$

Elimination of $z/(s\sqrt{2\varepsilon})$ on both sides by means of (2.18) yields the claim. $\square$

Let us make the change of variables $z\to s$ in (2.12), using (2.17) and the identity $y^2-z^2 = y^2\frac{1-2\varepsilon s^2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}$. It yields

$$\begin{aligned}
H_\varepsilon(\psi) &= \frac{1}{4\sqrt{2}\varepsilon^{\frac{3}{2}}}\int_{S^2\times S^2}\int_{y=0}^{\infty}\int_{z=-y}^{y}[a][\psi](y^2-z^2)\big(1+\cos^2\theta\big) \\
&\quad\times e^{-\frac{1}{2\varepsilon}\frac{z^2}{z^2+(y^2-z^2)\sin^2\theta/2}}e^{-\frac{\varepsilon}{4}\left(z^2+(y^2-z^2)\sin^2\theta/2\right)}e^{y/\sqrt{2}}dy\frac{dz}{\sqrt{z^2(1-\sin^2\theta/2)+y^2\sin^2\theta/2}}d\mathbf{\Omega}'d\mathbf{\Omega} \\
&= \frac{1}{4\varepsilon}\int_{S^2\times S^2}\int_{y=0}^{\infty}\int_{s=-\frac{1}{\sqrt{2\varepsilon}}}^{\frac{1}{\sqrt{2\varepsilon}}}[a][\psi]y^2\frac{1-2\varepsilon s^2}{\big(1-2\varepsilon s^2(1-\sin^2\theta/2)\big)^2}\big(1+\cos^2\theta\big) \\
&\qquad\qquad\times e^{-s^2}e^{-\frac{\varepsilon}{4}\frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}}e^{y/\sqrt{2}}dsdyd\mathbf{\Omega}'d\mathbf{\Omega}.
\end{aligned} \tag{2.19}$$

We claim that this expression (2.19) is a convenient formulation for passing to the limit $\varepsilon\to 0^+$. Indeed the strong singularity $|\omega|^{-1}$ is replaced by a mild singularity where the main term is $\frac{e^{-s^2}}{1-2\varepsilon s^2(1-\sin^2\theta/2)}$ for $-\frac{1}{\sqrt{2\varepsilon}}<s<\frac{1}{\sqrt{2\varepsilon}}$. To continue the discussion, we consider that the function f is isotropic, that is independent of $\mathbf{\Omega}$. In this case $[a]$ and $[\psi]$ in (2.19) can be roughly expressed as a difference as in $[g]=g(\nu)-g(\nu')=(\nu-\nu')\partial_\nu g(\nu)+O(\nu-\nu')^2$. In view of the change of variables (2.10) and (2.15), one gets a coefficient $\sqrt{\varepsilon}$. Since this algebra is done for $[a]$ and $[\psi]$ one gets a coefficient $\varepsilon=(\sqrt{\varepsilon})^2$ which is counterbalanced by the $\frac{1}{\varepsilon}$ in front of $H_\varepsilon(\psi)$ in (2.19). This regime corresponds to the main physical one. Moreover it simplifies the mathematical analysis.

2.2 Passing to the limit for isotropic profiles

One can easily checks that isotropic functions f are preserved by the collision kernel of the non dimensional Boltzman equation (2.2). Focusing on isotropic profiles is a way to get rid of fast angular damping effects and to consider frequency damping which is expected to occur at longer time scales. We therefore restrict the study to isotropic profiles. The restriction is only in this Section and it simplifies a lot the technicalities.

Lemma 4. *Assume that f and ψ are independent of $\mathbf{\Omega}$. One gets the expression*

$$\begin{aligned}
H_\varepsilon(\psi) &= -\frac{2\pi^2}{\varepsilon}\int_{y=0}^{\infty}\int_{s=-\frac{1}{\sqrt{2\varepsilon}}}^{\frac{1}{\sqrt{2\varepsilon}}}k(\nu,\nu')[b][\psi]y^2e^{-s^2}e^{-y/\sqrt{2}} \\
&\qquad\times\int_{\theta=0}^{\pi}e^{-\frac{\varepsilon}{4}\frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\frac{(1+\cos^2\theta)(1-2\varepsilon s^2)}{\big(1-2\varepsilon s^2(1-\sin^2\theta/2)\big)^2}\sin\theta d\theta dsdy
\end{aligned} \tag{2.20}$$

where $k(\nu,\nu')=(1+f(\nu))(1+f(\nu'))$, $[b](\nu,\nu')=b(\nu)-b(\nu')$ with $b(\nu)=e^\nu f(\nu)/(1+f(\nu))$.

Proof. Plug the isotropic Ansatz into (2.19) and simplify the angular integration by using (2.7) We used the identity $[a] = -k(\nu,\nu')[b]e^{-\nu-\nu'} = -k(\nu,\nu')[b]e^{-\sqrt{2}y}$. The proof is ended. $\square$

The next step is to expand $[b]$ and $[\psi]$ with respect to ε. Indeed the physical variables are expressed with respect to y and s as

$$\nu = y/\sqrt{2}+\frac{|\sin\theta/2|}{\sqrt{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\sqrt{\varepsilon}ys \text{ and } \nu' = y/\sqrt{2}-\frac{|\sin\theta/2|}{\sqrt{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\sqrt{\varepsilon}ys. \quad (2.21)$$

It can be used directly to determine the first order contribution in $\frac{[\psi]}{\sqrt{\varepsilon}} = \frac{\psi(\nu)-\psi(\nu')}{\sqrt{\varepsilon}}$. Since the test function ψ is smooth and with compact support, there exists a constant $c(\psi)>0$ such that

$$\left|\frac{[\psi]}{\sqrt{\varepsilon}} - 2\frac{|\sin\theta/2|}{\sqrt{1-2\varepsilon s^2(1-\sin^2\theta/2)}}ys\partial_\nu\psi(y/\sqrt{2})\right| \le c(\psi)\sqrt{\varepsilon}y^2s^2 \quad (2.22)$$

where we used the bound (2.16). Similarly making the regularity assumption that $b\in W^{2,\infty}(\mathbb{R}^+,S^2)$, there exists a constant denoted as $c(b)>0$ such that

$$\left|\frac{[b]}{\sqrt{\varepsilon}} - 2\frac{|\sin\theta/2|}{\sqrt{1-2\varepsilon s^2(1-\sin^2\theta/2)}}ys\partial_\nu b(y/\sqrt{2})\right| \le c(b)\sqrt{\varepsilon}y^2s^2. \quad (2.23)$$

So it is natural to replace $\frac{1}{\varepsilon}k(\nu,\nu')[b][\psi]$ in $H_\varepsilon(\psi)$ by their formal limit, so as to define

$$\begin{aligned} I_\varepsilon(\psi) = -8\pi^2 \int_{y=0}^{\infty}\int_{s=-\frac{1}{\sqrt{2\varepsilon}}}^{\frac{1}{\sqrt{2\varepsilon}}} k\left(\frac{y}{\sqrt{2}},\frac{y}{\sqrt{2}}\right)\partial_\nu b\left(\frac{y}{\sqrt{2}}\right)\partial_\nu\psi\left(\frac{y}{\sqrt{2}}\right) \\ \times y^4s^2e^{-s^2}e^{-y/\sqrt{2}}\int_{\theta=0}^{\pi} e^{-\frac{\varepsilon}{4}\frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\frac{(1+\cos^2\theta)(1-2\varepsilon s^2)\sin^2\theta/2}{\left(1-2\varepsilon s^2(1-\sin^2\theta/2)\right)^3}\sin\theta d\theta ds dy. \end{aligned} \quad (2.24)$$

Finally we replace the weight $e^{-\frac{\varepsilon}{4}\frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\frac{(1+\cos^2\theta)(1-2\varepsilon s^2)\sin^2\theta/2}{(1-2\varepsilon s^2(1-\sin^2\theta/2))^3}$ in the second line by its formal limit which is $(1+\cos^2\theta)\sin^2\theta/2$. It defines $L_\varepsilon(\psi)$ which still depends on ε through the bounds of the integral

$$\begin{aligned} L_\varepsilon(\psi) = -4\pi \int_{y=0}^{\infty}\int_{s=-\frac{1}{\sqrt{2\varepsilon}}}^{\frac{1}{\sqrt{2\varepsilon}}} k\left(\frac{y}{\sqrt{2}},\frac{y}{\sqrt{2}}\right)\partial_\nu b\left(\frac{y}{\sqrt{2}}\right)\partial_\nu\psi\left(\frac{y}{\sqrt{2}}\right) \\ \times y^4s^2e^{-s^2}e^{-y/\sqrt{2}}\int_{\theta=0}^{\pi}(1+\cos^2\theta)\sin^2\theta/2\sin\theta d\theta ds dy. \end{aligned} \quad (2.25)$$

For convenience, the proof that $H_\varepsilon(\psi)$ tends to $L_0(\psi)$ is decomposed in different steps.

Lemma 5. *Assume $f\in W^{1,\infty}(\mathbb{R}^+)$ is such that $b\in W^{2,\infty}(\mathbb{R}^+)$. There exists a constant denoted as $c(f,\psi)>0$ such that $|H_\varepsilon(\psi)-I_\varepsilon(\psi)|\le c(f,\psi)\sqrt{\varepsilon}$.*

Proof. In view of (2.21) and of the hypothesis, one has the inequality

$$\left|k(\nu,\nu') - k\left(\frac{y}{\sqrt{2}},\frac{y}{\sqrt{2}}\right)\right| \le c(f)\sqrt{\varepsilon}ys. \quad (2.26)$$

In combination with (2.22-2.23), it yields for some constant $C>0$

$$\begin{aligned} \left|k(\nu,\nu')\frac{[b]}{\sqrt{\varepsilon}}\frac{[\psi]}{\sqrt{\varepsilon}} - k\left(\frac{y}{\sqrt{2}},\frac{y}{\sqrt{2}}\right)\left(2\frac{|\sin\theta/2|}{\sqrt{1-2\varepsilon s^2(1-\sin^2\theta/2)}}ys\right)^2\partial_\nu b(y/\sqrt{2})\partial_\nu\psi(y/\sqrt{2})\right| \\ \le C\left(ys+y^2s^2\right)\sqrt{\varepsilon}. \end{aligned}$$

Therefore

$$|H_\varepsilon(\psi) - I_\varepsilon(\psi)| \leq C\sqrt{\varepsilon}\int_{y=0}^{\infty}\int_{s=-\frac{1}{\sqrt{2\varepsilon}}}^{\frac{1}{\sqrt{2\varepsilon}}}\int_{\theta=0}^{\pi}(ys + y^2s^2)y^2e^{-s^2}e^{-y/\sqrt{2}}$$
$$\times\, e^{-\frac{\varepsilon}{4}\frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\frac{(1+\cos^2\theta)(1-2\varepsilon s^2)}{\big(1-2\varepsilon s^2(1-\sin^2\theta/2)\big)^2}\sin\theta d\theta ds dy$$

By posing $\mu = -\cos\theta$ one has

$$\int_0^\pi \frac{1-2\varepsilon s^2\sin\theta d\theta}{1-2\varepsilon s^2(1-\sin^2\theta/2)^2} = 2\int_0^1 \frac{1-2\varepsilon s^2 d\mu}{(1-2\varepsilon s^2\frac{1+\mu}{2})^2} = 2\frac{1-2\varepsilon s^2}{\varepsilon s^2}\left(\frac{\varepsilon s^2}{(1-2\varepsilon s^2)(1-\varepsilon s^2)}\right) \leq 4$$

Using $1+\cos\theta^2 \leq 2$, it shows that

$$|H_\varepsilon(\psi) - I_\varepsilon(\psi)| \leq 8C\sqrt{\varepsilon}\int_{y=0}^{\infty}\int_{s=-\frac{1}{\sqrt{2\varepsilon}}}^{\frac{1}{\sqrt{2\varepsilon}}}(ys + y^2s^2)y^2e^{-s^2}e^{-y/\sqrt{2}}dsdy. \tag{2.27}$$

This double integral can be bounded by separation of variables. It yields the product of an integral in y which is immediately bounded times integrals with respect to s which can be reduced to

$$I_p(\varepsilon) = \int_0^{\frac{1}{\sqrt{2\varepsilon}}} s^p e^{-s^2}ds \leq \int_0^\infty s^p e^{-s^2}ds := C_p,$$

and $C_p > 0$ is independent of $\varepsilon > 0$. Plugging in (2.27) yields the claim. □

Lemma 6. *Assume $f \in L^\infty(\mathbb{R}^+)$ is such that $b \in W^{1,\infty}(\mathbb{R}^+)$. There exists a constant denoted as $k > 0$ such that $|I_\varepsilon(\psi) - L_\varepsilon(\psi)| \leq k\varepsilon$.*

Proof. It is the same method as in the proof of the previous Lemma. One expresses the difference $e^{-\frac{\varepsilon}{4}\frac{y^2\sin^2\theta/2}{1-2\varepsilon s^2(1-\sin^2\theta/2)}}\frac{(1-2\varepsilon s^2)}{(1-2\varepsilon s^2(1-\sin^2\theta/2))^3} - 1$ as a telescopic sum. It makes appear a linear dependence with respect to ε, multiplied by various terms which are ultimately bounded by estimates similar as in the previous Lemma. □

One can now pass to the limit $\varepsilon \to 0^+$ in (2.20) in $H_\varepsilon(\psi)$.

Proposition 7. *Assume $f \in W^{1,\infty}(\mathbb{R}^+)$ is such that $b \in W^{2,\infty}(\mathbb{R}^+)$. There exists a universal constant $C_\mathrm{k} > 0$ such that*

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(\psi) = L_0(\psi) = -C_\mathrm{k}\int_{y=0}^\infty k\,(\nu,\nu)\,\partial_\nu b\,(\nu)\,\partial_\nu\psi\,(\nu)\,\nu^4 e^{-\nu}d\nu.$$

Proof. One has directly that $L_\varepsilon(\psi) \to L_0(\psi)$ which is computed by separation of variables and a change of variable $\nu = \frac{y}{\sqrt{2}}$. The constant comes from the integral with respect to s and θ, and is given by

$$C_\mathrm{k} = 4\pi\int_{-\infty}^{\infty} s^2e^{-s^2}\int_{\theta=0}^{\pi}(1+\cos^2\theta)\sin^2\theta/2\sin\theta d\theta ds = \frac{8\pi^{3/2}}{3}.$$

□

Theorem 8. *Assume $f \in W^{1,\infty}(\mathbb{R}^+)$ is isotropic and such that $b \in W^{2,\infty}(\mathbb{R}^+)$. Then*

$$\frac{1}{\varepsilon^{\frac{3}{2}}\nu^2}\int_{\mathbb{R}^+\times S^2}\nu'\nu\big(1+\cos^2\theta\big)B_\varepsilon(\nu,\nu',\theta)q(f)d\nu' d\mathbf{\Omega}' \longrightarrow C_\mathrm{k}\frac{\partial}{\partial\nu}\left(\nu^4\left(\frac{\partial}{\partial\nu}f + (1+f)f\right)\right)$$

in the sense of distributions.

Proof. This is performed by integration by parts from the previous proposition and the fact that

$$e^{-\nu}k\left(\nu,\nu\right)\partial_\nu b = e^{-\nu}(1+f)^2\partial_\nu\left(e^\nu f/(1+f)\right) = f(1+f)+\partial_\nu f.$$

The proof is ended. □

It is convenient for the physical interpretation to reintroduce a posteriori the temperature of electrons $T>0$ in the equation and to renormalize the scattering coefficient in other to get rid of the constant C_k. We obtain $\partial_t f = \sigma_s \nu^{-2}\frac{\partial}{\partial\nu}\left(\nu^4\left(T\partial\nu f+(1+f)f\right)\right)$.

2.3 Passing to the limit for anisotropic profiles

We show now that, for anisotropic distributions f, the leading order term of $Q_\varepsilon(f)$ is a Lorentz operator.

Theorem 9. *Assume $f\in W^{1,\infty}(\mathbb{R}^+)$ is isotropic and such that $b\in W^{2,\infty}(\mathbb{R}^+)$. Then*

$$\varepsilon Q_\varepsilon(f) \longrightarrow = \frac{\sqrt{\pi}}{2}\int_{S^2}\left(1+\cos^2\theta\right)\left(f'-f\right)d\mathbf{\Omega}'$$

in the sense of distributions.

Proof. Using (2.19) with the change of variable $r=\sqrt{\varepsilon}s$ and Fubini theorem, one can write

$$\varepsilon\int_{\mathbb{R}^+\times S^2} Q_\varepsilon(f)\psi(\nu,\mathbf{\Omega})d\nu d\mathbf{\Omega} = \varepsilon H_\varepsilon(\phi) = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\frac{1}{\varepsilon^{\frac{1}{2}}}\exp(\frac{-r^2}{\varepsilon})g_\varepsilon(r)dr. \tag{2.28}$$

and g is a continuous fonction of r. As it is well known, in the sense of distribution $\frac{1}{\varepsilon^{\frac{1}{2}}}\exp(\frac{-r^2}{\varepsilon})\to\sqrt{\pi}\delta_0$. Thus

$$\lim_{\varepsilon\to 0}\varepsilon H_\varepsilon = \sqrt{\pi}g_0(0) = \frac{\sqrt{\pi}}{4}\int_{S^2\times S^2\times\mathbb{R}^+}(1+\cos\theta^2)(f(\nu,\mathbf{\Omega}')-f(\nu,\mathbf{\Omega}))(\psi(\nu,\mathbf{\Omega}')-\psi(\nu,\mathbf{\Omega}))d\nu d\mathbf{\Omega}d\mathbf{\Omega}'$$

$$= \frac{\sqrt{\pi}}{2}\int_{S^2\times S^2\times\mathbb{R}^+}(1+\cos\theta^2)(f(\nu,\mathbf{\Omega}')-f(\nu,\mathbf{\Omega}))\psi(\nu,\mathbf{\Omega}'))d\nu d\mathbf{\Omega}d\mathbf{\Omega}'$$

which is the desired result. □

3 The Pomraning model

The Pomraning anisotropic model (1.7) was obtained in [POM73] by cancelling the angular dependance in the energy exchange terms in a Fokker-Planck expansion. For the simplicity we consider the Pomraning model from now on with non dimensional variables similar to what was use in the previous section

$$\partial_t f = \frac{\sigma_s}{\varepsilon}\int_{S^2}\left(1+\cos^2\theta\right)\left(f'-f\right)d\mathbf{\Omega}' + \sigma_s\nu^{-2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T\partial_\nu f + f\left(1+f\right)\right)\right]. \tag{3.1}$$

We systematically use the convention that the measure over the sphere is normalized $\int_{S^2}d\mathbf{\Omega}=1$. The right hand side is the sum of the classical Thomson scattering which clearly reintroduces angular damping effects plus a Kompaneets term. In this part we prove several theoretical results for this equation, such as the conservation of the non negativity of the distribution function (lemma 10) and a H-theorem (lemma 11). With these results, equation (1.7) owns all the properties of the Boltzmann equation expressed in the introduction, properties that we wanted to keep while deriving simplified models. The end of this section deals with the proof of several mathematical properties of this model. The most important one is the proof of a H-theorem, which ensures the growth of the physical entropy of the model and thus the non reversibility of the process. We need to make the following assumptions

- $(\overline{\text{H1}})$ Initial conditions: the distribution function $f_0 = f(t=0)$ is non negative.
- $(\overline{\text{H2}})$ The scattering coefficient σ_s is non negative.

The equation (1.7) inherits of some of the properties of the equation (1.1). In particular it preserves the total number of photons and the stationary states (1.6). Since the H-theorem uses the logarithm of the solution of (1.7), one needs to prove that it remains non negative. This is done in the

Lemma 10 (Non negativity)**.** *Under assumptions* $(\overline{H1}) - (\overline{H2})$*, equation* (1.7) *preserves the non negativity of the distribution function, i.e.* $f_0(\nu,\mathbf{\Omega}) \geq 0 \Longrightarrow f(t,\nu,\mathbf{\Omega}) \geq 0$*, for all* $0 < t < T$*.*

Proof. We mainly follow the proof of Carrillo et al [CRS08], that is to introduce a regularized monotone increasing approximation sgn_ε of the sgn function and to define a regularized approximation $|f|_\varepsilon$ of the absolute value function $|f|$ via the primitive of $\text{sgn}_\varepsilon(f)$:

$$|f|_\varepsilon = \int_0^f \text{sgn}_\varepsilon(g)dg. \tag{3.2}$$

We also define a regularized approximation of the negative part of a function f as $f_\varepsilon^- = (|f|_\varepsilon - f)/2$. Multiplying equation (1.7) by $\nu^2\text{sgn}_\varepsilon(f)$ and integrating over $\mathbb{R}^+_\nu \times S^2$, one has

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^+\times S^2} |f|_\varepsilon \nu^2 d\nu d\mathbf{\Omega} = \frac{\sigma_s}{\varepsilon}\int_{\mathbb{R}^+\times S^2} &\text{sgn}_\varepsilon(f)\big(1+\cos^2\theta\big)\Big(f'-f\Big)\nu^2 d\mathbf{\Omega}' d\mathbf{\Omega} d\nu \\ + \sigma_s \int_{\mathbb{R}^+\times S^2} &\text{sgn}_\varepsilon(f)\frac{\partial}{\partial\nu}\left[\nu^4\Big(T\partial_\nu f + f\big(1+f\big)\Big)\right] d\nu d\mathbf{\Omega}.\end{aligned}$$

The conservation of the total number of photons

$$\frac{d}{dt}\int_{\mathbb{R}^+\times S^2} f\nu^2 d\nu d\mathbf{\Omega} = 0 = \frac{d}{dt}\int_{\mathbb{R}^+\times S^2}\big(|f|_\varepsilon - 2f_\varepsilon^-\big)\nu^2 d\nu d\mathbf{\Omega}$$

leads in particular to

$$\begin{aligned}2\frac{d}{dt}\int_{\mathbb{R}^+\times S^2} f_\varepsilon^- \nu^2 d\nu d\mathbf{\Omega} = \sigma_s\int_{\mathbb{R}^+\times S^2} &\text{sgn}_\varepsilon(f)\big(1+\cos^2\theta\big)\Big(f'-f\Big)\nu^2 d\mathbf{\Omega}' d\mathbf{\Omega} d\nu \\ + \sigma_s \int_{\mathbb{R}^+\times S^2} &\text{sgn}_\varepsilon(f)\frac{\partial}{\partial\nu}\left[\nu^4\Big(T\partial_\nu f + f\big(1+f\big)\Big)\right] d\mathbf{\Omega} d\nu.\end{aligned}$$

Let us denote P_1 the first term of the right hand side and P_2 the second one. The invariance by change of variable $\mathbf{\Omega} \to \mathbf{\Omega}'$ of $\cos\theta$ yields

$$P_1 = -\frac{\sigma_s}{\varepsilon}\int_{\mathbb{R}^+\times S^2}\big(1+\cos^2\theta\big)\big(f'-f\big)\Big(\text{sgn}_\varepsilon(f') - \text{sgn}_\varepsilon(f)\Big)\nu^2 d\mathbf{\Omega}' d\mathbf{\Omega} d\nu.$$

The non decreasing monotonicity of the function $\text{sgn}_\varepsilon(f)$ thus yields $P_1 \leq 0$. For the term P_2, one has, using an integration by parts

$$P_2 = -\sigma_s\int_{\mathbb{R}^+\times S^2}\nu^4\ \text{sgn}_\varepsilon'(f)\Big(T|\partial_\nu f|^2 + f\big(1+f\big)\partial_\nu f\Big) d\mathbf{\Omega} d\nu.$$

As pointed out by Carrillo et al, one has $\text{sgn}_\varepsilon'(f) f\partial_\nu f = \partial_\nu\big(f\text{sgn}_\varepsilon(f) - |f|_\varepsilon\big)$ and $\text{sgn}_\varepsilon'(f)f^2\partial_\nu f = \partial_\nu\big(f^2 sgn_\varepsilon(f) - f|f|_\varepsilon\big)$. Passing to the limit as $\varepsilon \to 0$, it yields $P_2 \leq 0$ and finally

$$\int_{\mathbb{R}^+\times S^2} f^-(t)\nu^2 d\nu d\mathbf{\Omega} \leq \int_{\mathbb{R}^+\times S^2} f^-(0)\nu^2 d\nu d\mathbf{\Omega} = 0,$$

which ends the proof. □

Lemma 11 (H-Theorem)**.** *Assume that assumptions* $(\overline{H1}) - (\overline{H2})$ *are satisfied and consider the function* $\phi(f) = f\log(f) - (f+1)\log(f+1) + \frac{\nu}{T}f$*. The following inequality holds*

$$H'(t) = \frac{d}{dt}\int_{\mathbb{R}^+\times S^2} \phi(f)\nu^2 d\nu d\mathbf{\Omega} \le 0.$$

Proof. The function ϕ is the sum of the (mathematical) entropy of the photons $f\log(f) - (f+1)\log(f+1)$ and a term $\frac{\nu}{T}f$ which is the entropy of the electrons. Let us decompose equation (3.1) as $\partial_t f = P_{sym} + P_{komp}$, where

$$\begin{cases} P_{sym} = \dfrac{\sigma_s}{\varepsilon}\displaystyle\int_{S^2}\left(1+\cos^2\theta\right)\Big(f' - f\Big)d\mathbf{\Omega}' \\ \\ P_{komp} = \sigma_s \nu^{-2}\dfrac{\partial}{\partial\nu}\left[\nu^4\Big(T\partial_\nu f + f(1+f)\Big)\right]. \end{cases}$$

It yields, by definition of the entropy H, $H'(t) = \int_{\mathbb{R}^+\times S^2}\Big(P_{sym} + P_{komp}\Big)\phi'(f)\nu^2 d\nu d\mathbf{\Omega}$. We thus have to estimate two terms. For the first one, one has by definition of P_{sym}

$$\int_{\mathbb{R}^+\times S^2} P_{sym}\phi'(f)\nu^2 d\nu d\mathbf{\Omega} = \frac{\sigma_s}{\varepsilon}\int_{\mathbb{R}^+\times S^2}\left(1+\cos^2\theta\right)\Big(f'-f\Big)\Bigg(\log\left(\frac{f}{1+f}\right) + \frac{\nu}{T}\Bigg)\nu^2 d\nu d\mathbf{\Omega}d\mathbf{\Omega}'.$$

Simple arguments yield $\int_{\mathbb{R}^+\times S^2}\left(1+\cos^2\theta\right)\Big(f'-f\Big)\frac{\nu^3}{T}d\nu d\mathbf{\Omega}d\mathbf{\Omega}' = 0$. Using the invariance by change of variable $\mathbf{\Omega}\to\mathbf{\Omega}'$ of $\cos\theta = \mathbf{\Omega}.\mathbf{\Omega}'$, one can write the remaining term as

$$\int_{\mathbb{R}^+\times S^2} P_{sym}\phi'(f)\nu^2 d\nu d\mathbf{\Omega} = -\frac{\sigma_s}{\varepsilon}\int_{\mathbb{R}^+\times S^2}\left(1+\cos^2\theta\right)\Big(f - f'\Big) \\ \times\left\{\log\left(\frac{f}{1+f}\right) - \log\left(\frac{f'}{1+f'}\right)\right\}\nu^2 d\nu d\mathbf{\Omega}d\mathbf{\Omega}'.$$

The monotone increasing behavior of the function $X\mapsto\log\left(\frac{X}{1+X}\right)$ and the non negativity of the distribution function (lemma 10) yield

$$\int_{\mathbb{R}^+\times S^2} P_{sym}\phi'(f)\nu^2 d\nu d\mathbf{\Omega} \le 0.$$

We now turn to the Kompaneets type term P_{komp}. This term has already been studied for a slightly different Fokker-Planck equation in [CRS08] (see also [BC03]). One has, using an integration by parts,

$$\int_{\mathbb{R}^+\times S^2} P_{komp}\phi'(f)\nu^2 d\nu d\mathbf{\Omega} = -\sigma_s\int_{\mathbb{R}^+\times S^2}\left(\frac{\partial_\nu f}{f(f+1)} + \frac{1}{T}\right)\nu^4\Big(T\partial_\nu f + f(1+f)\Big)d\nu d\mathbf{\Omega}.$$

It thus yields

$$\int_{\mathbb{R}^+\times S^2} P_{komp}\phi'(f)\nu^2 d\nu d\mathbf{\Omega} = -\sigma_s\int_{\mathbb{R}^+\times S^2}\frac{T}{f(f+1)}\nu^4\left(\partial_\nu f + \frac{f(1+f)}{T}\right)^2 d\nu d\mathbf{\Omega}.$$

Once again, the non negativity of the distribution function gives $\int P_{komp}\phi'(f)\nu^2 d\nu d\mathbf{\Omega} \le 0$. The proof is concluded. ∎

Finally, we prove a comparison result.

Lemma 12 (Comparison principle)**.** *Assume that f and g are two solutions of* (1.7) *that satisfied the assumptions* $\overline{H}1$-$\overline{H}2$ *and* $f_0 \geq g_0$*. Then* $f(t) \geq g(t)$ *for all* $t \in [0, T]$*.*

Proof. Once again, we follow [CRS08]. Since f and g are both solutions of (1.7), one can write the equation satisfies by $h = f - g$:

$$\partial_t h = \frac{\sigma_s}{\varepsilon} \int_{S^2} \left(1 + \cos^2\theta\right) \Big(h' - h\Big) d\mathbf{\Omega}' + \sigma_s \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \Big(T\partial_\nu h + h + h\big(f + g\big)\Big)\right],$$

Multiplying this equation by $sgn_\varepsilon(h)\nu^2$ and using the conservation of the total number of photons, one gets, with the same notations than in the previous lemma (positivity, lemma 10),

$$\frac{d}{dt} \int h_\varepsilon^- \nu^2 d\nu d\mathbf{\Omega} = \frac{\sigma_s}{\varepsilon} \int \mathrm{sgn}_\varepsilon(h) \nu^2 \left(1 + \cos^2\theta\right) \Big(h' - h\Big) d\mathbf{\Omega}'$$
$$+ \sigma_s \int \mathrm{sgn}_\varepsilon(h) \frac{\partial}{\partial \nu} \left[\nu^4 \Big(T\partial_\nu h + h + h\big(f + g\big)\Big)\right].$$

The relation $sgn'_\varepsilon(h) h(f+g)\partial_\nu h = (f+g)\partial_\nu \big(h\mathrm{sgn}_\varepsilon(h) - |h|_\varepsilon\big)$ together with the previous procedure (lemma 10) gives

$$\int h_\varepsilon^-(t)\nu^2 d\nu d\mathbf{\Omega} \leq \int h_\varepsilon^-(0)\nu^2 d\nu d\mathbf{\Omega} = 0$$

thanks to the initial conditions, and this gives the announced result. □

4 Angular moment models

What we call an angular or anisotropic moment model in this section is any simplified anisotropic system of equations which captures some flavor of the limit isotropic Kompaneets equation. We consider two different methods for the design of such models. The first one, called M_1 approximation, is based on the symmetrized kernels with full integrations (2.12) or (2.20). Even if quite rigorous, it produces fully non linear models and this viewed as a disadvantage in view of a numerical study. On the contrary the P_1 approximation method (used for example in the recent works [BRU00, BRU02, HLM10]) yields a linear-quadratic model, at the price of a more phenomenological derivation. The P_1 approximation has many interesting features. Its structure is a system of two quadratic Fokker-Planck equations which can be studied in order to obtain a comparison with the Kompaneets equation. In particular it is known [CL86] that in long time range, the solution of the Kompaneets equation might condensate near $\nu = 0$, depending on the initial number of photons. We show that, for the new P_1 approximation in the limit case where the Fokker-Planck term dominates the friction term in the first order moment equation, the value of the threshold on the number of photons is modified, i.e. the anisotropic part of the distribution introduces a new set of solutions. The result is easily proved once realizes that it is possible to decouple the P_1 system into two independent Kompaneets equations. Finally several numerical illustration conclude this behavior.

4.1 M_1 approximation

The integrals (2.12) and (2.20) can be a natural starting point for the development of anisotropic models. We just give the main ideas on one example. Apart from technical details, it is sufficient to take (2.20) and to insert an angular expansion. We consider the expansion

$$b(\nu, \mathbf{\Omega}) = b_0(\nu) + \sqrt{\varepsilon}\mathbf{b}_1(\nu) \cdot \mathbf{\Omega}.$$

Since $b = e^\nu f/(1 + f)$, it means that f is a generalized Bose function

$$f(\nu, \mathbf{\Omega}) = \frac{1}{e^{\nu b(\nu, \mathbf{\Omega})} - 1}.$$

This is very similar to moment models in transfer. The cornerstone is the quadratic form $\mathcal{A}_\varepsilon(b,\psi) = \frac{[b][\psi]}{\varepsilon}$ where we insert a test function with the same Ansatz $\psi(\nu,\mathbf{\Omega}) = c_0(\nu)+\sqrt{\varepsilon}\mathbf{c}_1(\nu)\cdot\mathbf{\Omega}$. Considering the calculations in (2.22-2.23), a natural approximation is $\mathcal{A}_\varepsilon(b,\psi) \approx \mathcal{A}[b_0,\mathbf{b}_1 : c_0,\mathbf{c}_1]$ where

$$\mathcal{A}[b_0,\mathbf{b}_1 : c_0,\mathbf{c}_1] = (2|\sin\theta/2|ys\partial_\nu b_0(\nu) + \mathbf{b}_1(\nu)\cdot[\mathbf{\Omega}]) \\ \times (2|\sin\theta/2|ys\partial_\nu c_0(\nu) + \mathbf{c}_1(\nu)\cdot[\mathbf{\Omega}])$$

and where we use $\nu = \left(\frac{y}{\sqrt{2}}\right)$. Plugging directly in (2.25) one gets the integral

$$J(b_0,\mathbf{b}_1 : c_0,\mathbf{c}_1) = -4\pi\int_{y=0}^{\infty}\int_{\mathbb{R}}\int_{\theta=0}^{\pi} k\left(\frac{y}{\sqrt{2}},\frac{y}{\sqrt{2}}\right)\mathcal{A}[b_0,\mathbf{b}_1 : c_0,\mathbf{c}_1] \\ \times y^4 s^2 e^{-s^2} e^{-y/\sqrt{2}}(1+\cos^2\theta)\sin^2\theta/2\sin\theta d\theta ds dy. \tag{4.1}$$

A weak form can be formally written as

$$\partial_t\int_{\mathbb{R}^+}\frac{c_0(\nu)+\mathbf{c}_1(\nu)\cdot\mathbf{\Omega}}{e^{\nu b_0(\nu)+\nu\mathbf{b}_1(\nu)\cdot\mathbf{\Omega}}-1}d\nu d\mathbf{\Omega} = J(b_0,\mathbf{b}_1 : c_0,\mathbf{c}_1) \tag{4.2}$$

for all test functions c_0 and $\mathbf{c}_1$.

The interest of such formulations is an immediate algebraic version of the H-Theorem obtained for $(c_0,\mathbf{c}_1) = (b_0,\mathbf{b}_1)$, since $J(b_0,\mathbf{b}_1 : b_0,\mathbf{b}_1) \leq 0$ by definition. A disadvantage is the non linearity that shows up under the time and space derivatives. The Pomraning model can probably be introduced as a further simplification of the bilinear form $\mathcal{A}$.

4.2 P_1 approximation

The P_1 approximation consists of assuming that the distribution function f writes

$$f(t,\nu,\mathbf{\Omega}) = f_0(t,\nu) + \mathbf{\Omega}.\mathbf{f_1}(t,\nu), \tag{4.3}$$

i.e. is affine with respect to $\mathbf{\Omega}$. The main advantages of the P_1 approximation is that the moments equations are easily constructed. Due to the polynomial angular dependence of f, an increase of the order of the approximation is easily achieved. Moreover and as we will see in this part, even if the source terms are nonlinear (quadratic in our case), they can be expressed in terms in the moments. The two first angular moments of the distribution function are

$$\begin{cases} E(t,\nu) = \displaystyle\int_{S^2} f(t,\nu,\mathbf{\Omega})d\mathbf{\Omega}, \\ \mathbf{F}(t,\nu) = \displaystyle\int_{S^2} f(t,\nu,\mathbf{\Omega})\mathbf{\Omega}d\mathbf{\Omega}, \end{cases} \tag{4.4}$$

where we remind the reader that the measure over the sphere is normalized $\int_{S^2} d\mathbf{\Omega} = 1$. The frequency dependent P_1 model is given in the following lemma.

Lemma 13 (Frequency dependent P_1 model)**.** *The frequency dependent P_1 approximation of the Pomraning model (3.1) is*

$$\begin{cases} \partial_t E = \sigma_s\nu^{-2}\dfrac{\partial}{\partial\nu}\left[\nu^4\Big(T\partial_\nu E + E + E^2 + 3(\mathbf{F},\mathbf{F})\Big)\right], \\ \partial_t\mathbf{F} = \sigma_s\nu^{-2}\dfrac{\partial}{\partial\nu}\left[\nu^4\Big(T\partial_\nu\mathbf{F} + \mathbf{F} + 2E\mathbf{F}\Big)\right] - \dfrac{4\sigma_s}{3}\mathbf{F}. \end{cases} \tag{4.5}$$

Proof. The definition of E and $\mathbf{F}$ together with the property $\int\mathbf{\Omega}d\mathbf{\Omega} = 0$ yields

$$f_0 = E \text{ and } \mathbf{f_1} = 3\mathbf{F}. \tag{4.6}$$

Integrating equation (3.1) over S^2 and dividing by 4π, we obtain the equation

$$\partial_t E = \sigma_s \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu E + E + \frac{1}{4\pi}\int_{S^2} f^2 d\mathbf{\Omega}\Big)\right].$$

Equations (4.3) and (4.6) gives

$$\int_{S^2} f^2 d\mathbf{\Omega} = E^2 + 3(\mathbf{F}, \mathbf{F}),$$

and thus the equation on E is

$$\partial_t E = \sigma_s \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu E + E + E^2 + 3(\mathbf{F}, \mathbf{F})\Big)\right]. \tag{4.7}$$

We now turn to the equation on $\mathbf{F}$. Multiplying equation of (1.7) by $\mathbf{\Omega}$ and integrating on S^2, one has

$$\partial_t \mathbf{F} = \sigma_s \int_{S^2\times S^2} (1+\cos^2\theta)\mathbf{\Omega}\big(f' - f\big) d\mathbf{\Omega}' d\mathbf{\Omega} \quad + \sigma_s \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu \mathbf{F} + \mathbf{F} + \frac{1}{4\pi}\int_{S^2} \mathbf{\Omega} f^2 d\mathbf{\Omega}\Big)\right].$$

Equations (4.3) and (4.6) give after some elementary calculus

$$\begin{cases} \displaystyle\int_{S^2\times S^2} (1+\cos^2\theta)\mathbf{\Omega}\big(f' - f\big) d\mathbf{\Omega}' d\mathbf{\Omega} = -\frac{4}{3}\mathbf{F}, \\ \displaystyle\int_{S^2} \mathbf{\Omega} f^2 d\mathbf{\Omega} = 2E\mathbf{F}, \end{cases}$$

which concludes the proof. □

It is easy to see that this P_1 model preserves the Bose-Einstein distributions f_μ (1.6). Indeed for such distributions one has $\mathbf{F} = \mathbf{0}$ and $E = f_\mu$, and these distributions satisfy $T\partial_\nu f_\mu + f_\mu + f_\mu^2 = 0$. From now on and until the end of this section, we take $c = 1$ in the transport term. The isotropic case $\mathbf{F} = 0$ can be seen as the P_0 model, and reduces to the Kompaneets equation

$$\partial_t f = \sigma_s \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu f + f(1+f)\Big)\right]. \tag{4.8}$$

We aim to study the differences between the Kompaneets equation and the P_0 model. To this end we use only on angular direction and consider that the friction damping $-\frac{4\sigma_s}{3}F$ is now phenomenological. It is written as $-\alpha F$ with $\alpha \geq 0$. This phenomenology can also be recovered through a convenient renormalization. It yields the system

$$\begin{cases} \partial_t E = \sigma_s \nu^{-2} \dfrac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu E + E + E^2 + 3F^2\Big)\right], \\ \partial_t F = \sigma_s \nu^{-2} \dfrac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu F + F + 2EF\Big)\right] - \alpha F. \end{cases} \tag{4.9}$$

4.3 Condensation of the P_1 model with vanishing friction

In view of Caflish-Levermore results [CL86] (4.8) recalled in the Appendix, the goal of this part is to understand how the anisotropic part of the radiation, modeled in (4.5) by the first moment $\mathbf{F}$, modifies the zero-th order moment E. On the one hand, it is clear that if $\mathbf{F}(t=0) = 0$, one has for all $t \geq 0$ $E(t) = f(t)$, where f is the solution of the Kompaneets equation. Moreover, if $\alpha >> 1$, one can expect $\mathbf{F}$ to tends quickly vers 0 and thus obtain $\lim_{t\to\infty} E(t) = f(t)$. One the other hand, the long time behavior of the solution of the P_1 model for $\alpha \approx 1$ or $\alpha << 1$ is not straightforward. This is the purpose of the forthcoming study. In particular we show in Theorem 14 that in the limit case $\alpha = 0$ there exists solutions of the P_1 model such that the number of photons at the initial stage is lower than the critical number $N(f_0)$ (A.2) and such that a Bose condensation nonetheless appears in long time range.

Let us emphasize some properties of the P_1 model. Firstly and as for the Kompaneets equation, the model (4.9) preserves the number of photons. Indeed, the number of photons is defined by

$$N(t) = \int_{\mathbb{R}^+ \times S^2} f(t, \nu, \mathbf{\Omega}) \nu^2 d\nu d\mathbf{\Omega}.$$

Since for the P_1 model $f = E + 3\mathbf{F}.\mathbf{\Omega}$, one easily obtains by using the P_1 model (4.9)

$$\frac{d}{dt} N(t) = \frac{d}{dt} \left[\int_{\mathbb{R}^+} E(t, \nu) \nu^2 d\nu \right] = 0,$$

assuming the correct flux conditions. What appears as a new property is the variation of the energy of the P_1 model in comparison with the P_0 model. The energy is defined by

$$E(t) = \int_{\mathbb{R}^+ \times S^2} f(t, \nu, \mathbf{\Omega}) \nu^3 d\nu d\mathbf{\Omega}.$$

Using the P_1 model (4.5) and an integration by parts, one gets

$$\frac{d}{dt} E(P_1) = \frac{d}{dt} \left[\int_{\mathbb{R}^+} E(t, \nu) \nu^3 d\nu \right] = - \int_{\mathbb{R}^+} \left(T \partial_\nu E + E + E^2 \right) \nu^4 d\nu - 3 \int_{\mathbb{R}^+} (\mathbf{F}, \mathbf{F}) \nu^4 d\nu.$$

This equation is recast as

$$\frac{d}{dt} E(P_1) = \frac{d}{dt} E(P_0) - 3 \int_{\mathbb{R}^+} (\mathbf{F}, \mathbf{F}) \nu^4 d\nu,$$

with an obvious definition of $E(P_0)$. This shows that the anisotropic part of the radiation does not modify the number of photons, but decreases the energy of the radiation.

We now turn to the main result of this part, which is concerned by the long time behavior of the solution of the P_1 model (4.9) in the case $\alpha = 0$. We prove the following lemma, which shows that the anisotropic part of the radiation exhibits different stationary solutions in comparison with the Kompaneets equation in long time range. In the following $N(f_0)$ still refers to the number of photons of the Bose-Einstein distribution with $\mu = 0$. This comes from the very simple remark that one can find a new set of variables, namely $Z^\pm = E \pm \sqrt{3} F$, which satisfies the Kompaneets equation. The next result is the consequence of this diagonalization.

Theorem 14 (Asymptotic behavior of the P_1 model in the case)**.** *Assume a zero friction parameter $\alpha = 0$. There exists solutions of the P_1 model* (4.9)*, whose total number of photons at the initial time is less than the critical number of photons $N(f_0)$ (A.2) , such that a condensation phenomena appears in long time range. For example if the initial conditions for the P_1 model* (4.9) *are chosen such that*

$$\begin{cases} N(E^{in}) = \dfrac{3}{4} N(f_0), \\ N(F^{in}) = \dfrac{1}{2\sqrt{3}} N(f_0), \end{cases}$$

(which yields $N(f^{in}) = N(E^{in} + 3\mathbf{\Omega}.F^{in}) = N(E^{in}) < N(f_0)$) then in long time range one has

$$\begin{cases} \lim\limits_{t \to \infty} E(t, .) = \nu^2 \dfrac{f_0 - f_\mu}{2} + \dfrac{N(f_0)}{8} \delta_0, \\ \lim\limits_{t \to \infty} F(t, .) = \nu^2 \dfrac{f_0 - f_\mu}{2\sqrt{3}} + \dfrac{N(f_0)}{8\sqrt{3}} \delta_0, \end{cases}$$

although $N(E^{in}) < N(f_0)$, where the parameter μ is such that $N(f_\mu) = N(f_0)/4$.

Proof. We set $Z^{\pm} = E \pm \sqrt{3}F$. Multiplying the second equation of (4.9) by $\sqrt{3}$ and adding and subtracting to the first equation, one sees that $Z^{\pm}$ satisfies the Kompaneets equation

$$\partial_t Z^{\pm} = \sigma_s \nu^{-2} \frac{\partial}{\partial \nu}\left[\nu^4 \Big(T\partial_\nu Z^{\pm} + Z^{\pm}\big(1+Z^{\pm}\big)\Big)\right].$$

The initial conditions for $Z^{\pm}$ are defined by $(Z^{\pm})^{in} = E^{in} \pm \sqrt{3}F^{in}$. Using the definition of E^{in} and F^{in}, one gets $N\big((Z^+)^{in}\big) = \frac{5}{4}N(f_0) > N(f_0)$. In the same way one obtains $N\big((Z^-)^{in}\big) = \frac{1}{4}N(f_0) < N(f_0)$. One applies the result A.1 on Z^+ and Z^-, and gets

$$\begin{cases} \lim\limits_{t\to\infty} \nu^2 Z^+(t,.) = \nu^2 f_0 + \dfrac{N(f_0)}{4}\delta_0, \\ \lim\limits_{t\to\infty} \nu^2 Z^-(t,.) = \nu^2 f_\mu, \quad \mu > 0 \text{ s. t. } N(f_\mu) = \dfrac{N(f_0)}{4}. \end{cases}$$

From the definition of $Z^{\pm}$, one has $E = \big(Z^+ + Z^-\big)/2$ and $F = \big(Z^+ - Z^-\big)/2\sqrt{3}$, which yields

$$\begin{cases} \lim\limits_{t\to\infty} E(t,.) = \nu^2 \dfrac{f_0 - f_\mu}{2} + \dfrac{N(f_0)}{8}\delta_0, \\ \lim\limits_{t\to\infty} F(t,.) = \nu^2 \dfrac{f_0 - f_\mu}{2\sqrt{3}} + \dfrac{N(f_0)}{8\sqrt{3}}\delta_0, \quad \mu > 0 \text{ s. t. } N(f_\mu) = \dfrac{N(f_0)}{4}, \end{cases}$$

which is the announced result. □

4.4 Numerical illustration

We present here a numerical scheme (see [BC03, LLPS84, DWLM09] for a literature on the topic) designed for the frequency dependent P_1 model (4.9) and we illustrate some theoretical results with numerical illustrations for different values of the friction parameter α. We take $\sigma_s = 1$ and $T = 1$.

4.4.1 A simple Finite Volume scheme

The frequency domain is $[0, \nu^*]$. For $1 \le j \le N$, we consider an irregular mesh defined by $(N+1)$ points $0 = \nu_{\frac{1}{2}} < ... < \nu_{N+\frac{1}{2}} = \nu^*$. We define ν_j as the middle of the j-th frequency band, i.e. $\nu_j = (\nu_{j-\frac{1}{2}} + \nu_{j+\frac{1}{2}})/2$ and we denote $\Delta\nu_j$ its length. We also define the dual $(j+\frac{1}{2})$-th frequency band as the cell $[\nu_j, \nu_{j+1}]$, which length is denoted $\Delta\nu_{j+\frac{1}{2}}$. Since we consider the homogeneous (in space) case, we consider the 1D case, and thus the first moment $\mathbf{F}$ is a scalar.

Due to the term ν^{-2} in front of the right hand side of the P_1 model (4.9), it appears to be easier to work on the variables $U = \nu^2 E$ and $V = \nu^2 F$. Using this set of variables, the frequency dependent P_1 model writes

$$\begin{cases} \partial_t U = \dfrac{\partial}{\partial \nu}\left[T\Big(\partial_\nu\big(\nu^2 U\big) - 4\nu U\Big) + U\big(\nu^2 + U\big) + 3V^2\Big)\right], \\ \partial_t V = \dfrac{\partial}{\partial \nu}\left[T\Big(\partial_\nu\big(\nu^2 V\big) - 4\nu V\Big) + V + 2UV\Big)\right] - \alpha V. \end{cases} \tag{4.10}$$

We use a classical finite volume scheme with explicit Euler discretization of the time derivatives, defined by

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} = \dfrac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta \nu_j}, \\ \dfrac{V_j^{n+1} - V_j^n}{\Delta t} = \dfrac{V_{j+\frac{1}{2}}^n - V_{j-\frac{1}{2}}^n}{\Delta \nu_j} - \alpha V_j^n, \end{cases} \tag{4.11}$$

where the fluxes are defined by

$$\begin{cases} U^n_{j+\frac12} = T\Big(\dfrac{\nu_{j+1}^2 U_{j+1}^n - \nu_j^2 U_j^n}{\Delta\nu_{j+\frac12}} - 4\nu_j U_j^n\Big) + U_{j+1}^n\big(\nu_{j+1}^2 + U_{j+1}^n\big) + 3\big(V_{j+1}^n, V_{j+1}^n\big), \\ V^n_{j+\frac12} = T\Big(\dfrac{\nu_{j+1}^2 V_{j+1}^n - \nu_j^2 V_j^n}{\Delta\nu_{j+\frac12}} - 4\nu_j V_j^n\Big) + V_{j+1}^n\big(\nu_{j+1}^2 + 2U_{j+1}^n\big). \end{cases} \tag{4.12}$$

Numerically, the conservation of the total number of photons is obtained by setting $U^n_{N+\frac12} = U^n_1 = 0$, for all n. To obtain the CFL condition, we write the system (4.10) as a drift diffusion system on the variable $\mathbf{W} = (U, V)$, that is

$$\partial_t \mathbf{W} = \nu^2 T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_\nu^2 \mathbf{W} + \underbrace{\begin{pmatrix} \nu^2 + 2U & 6V \\ 2V & \nu^2 + 2U \end{pmatrix}}_{A} \partial_\nu \mathbf{W} + \underbrace{\begin{pmatrix} 2(\nu - T) & 0 \\ 0 & 2(\nu - T) - \alpha \end{pmatrix}}_{R} \mathbf{W}. \tag{4.13}$$

Studying the eigenvalues of the matrix A and R, denoted respectively $\Lambda_A^\pm$ and $\Lambda_R^\pm$, one easily finds

$$\begin{cases} \Lambda_A^\pm = \nu^2 + 2U \pm 2\sqrt{3}V, \\ \Lambda_R^+ = 2(\nu - T), \\ \Lambda_R^- = 2(\nu - T) - \alpha, \end{cases}$$

The stability of the scheme is obtained under the following CFL condition

$$\Delta t \sup_{1\le j\le N} \left(\frac{T\nu_j^2}{\Delta\nu_j^2} + \frac{\max\big(|\Lambda_{A,j}^\pm|\big)}{\Delta\nu_j} + \max\big(|\Lambda_{R,j}^\pm|\big) \right) \le 1,$$

with obvious notations for $\Lambda_{A,j}^\pm$ and $\Lambda_{R,j}^\pm$.

4.4.2 Setting of the simulations

We compare the solution of the P_1 model with the solution of the P_0 model for several value of the friction coefficient α. All test cases are initialized with the same values, depicted in Figure 2 and we take for the electron temperature $T = 1$. The mesh is composed of 800 cells. With the choice of the variables (U, V), the number of photons at the initial stage is simply $\int_{\mathbb{R}^+} U d\nu = 1 \le N(f_0) \approx 2.4$. The long time behavior of the P_0 model is thus a regular Planck distribution, and our aim is to understand the influence of the first order moment V in the transitional and long time behavior of the zero-th order moment U. The numerical scheme is (4.11-4.12). The results are displayed at different times until a final time $T_f > 0$.

4.4.3 Strong friction $\alpha = 1$

We solve the P_1 system with $\alpha = 1$ and we compare the zero-th order moment to the solution of the P_0 equation.

Figure 3 displays the zero-th order moment U and the solution of the Kompaneets equation at different time. It shows that in the case $\alpha = 1$, the anisotropic part of the radiation does not sensibly modify the solution of the Kompaneets equation.

Figure 4 displays the evolution of the first order moment V. The friction term is dominant in comparison to the Fokker-Planck term.

Figure 5 presents the time evolution of the total energy of the P_0 and P_1 models. As expected (see equation (4.3)), the anisotropic part of the radiation decreases the energy, in comparison with the Kompaneets equation. Nevertheless, and since the first order moment V is small in comparison with the zero-th order moment, the variation in the total energy due to the anisotropic part is negligible.

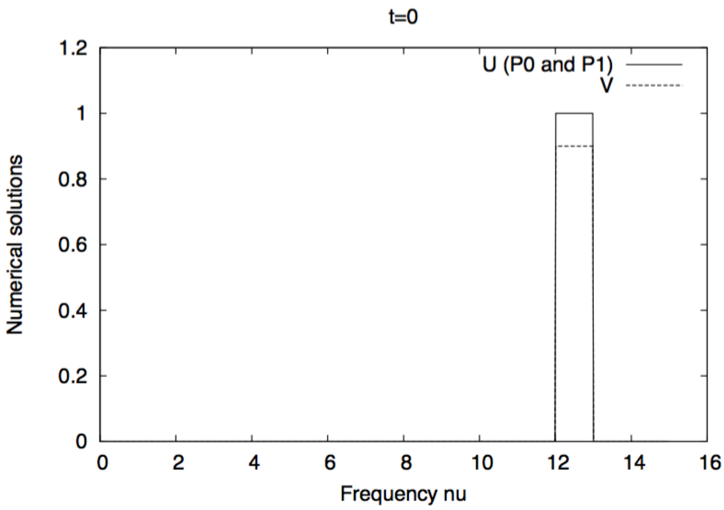

Figure 2: Initialization of the P_0 (Kompaneets) and P_1 model.

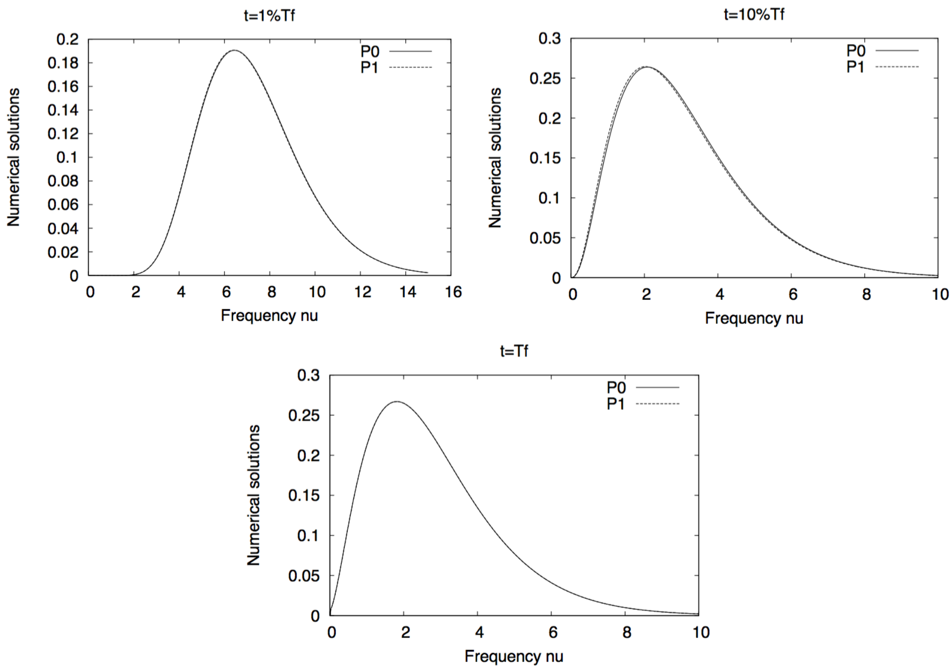

Figure 3: Zero-th order moments (P_0 and P_1) versus frequency ν at different times, $\alpha = 1$. The curves are almost merged, as expected due to the strong relaxation. The P_0 and P_1 models converge toward the same Planck distribution.

4.4.4 Mild friction $\alpha = 0.1$

In this part we perform the same study than in the previous part, but we take a smaller value of the friction coefficient $\alpha = 0.1$.

Figure 6 displays the zero-th order moment U and the solution of the Kompaneets equation, at different times. It shows in particular a different transitional regime. Contrary to the solution

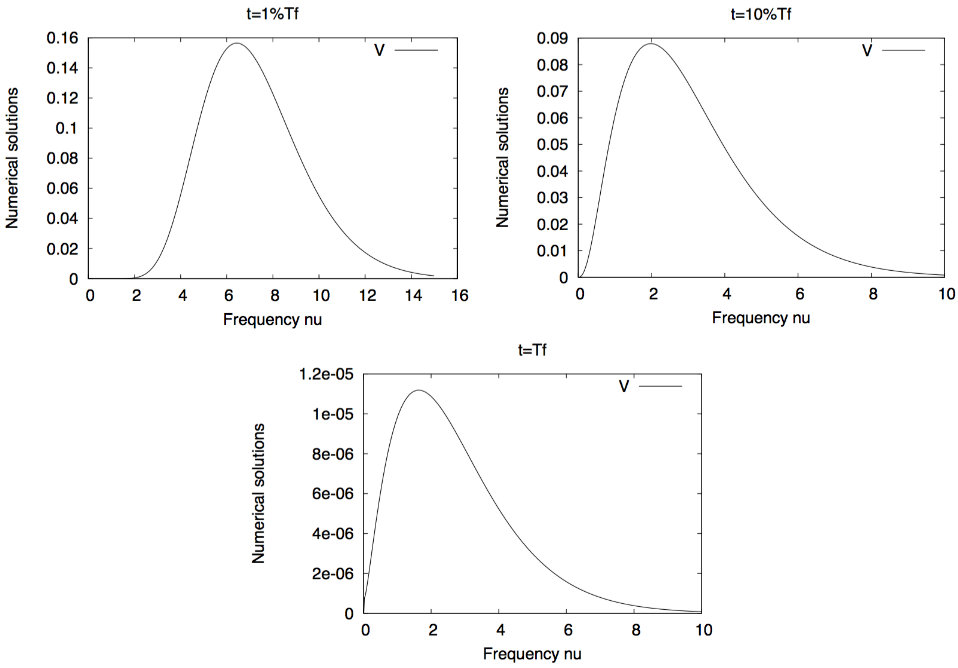

Figure 4: First order moment V versus frequency ν at different times, $\alpha = 1$. The first order moment has a Planck profile, decreasing with time due to the relaxation ($\alpha = 1$).

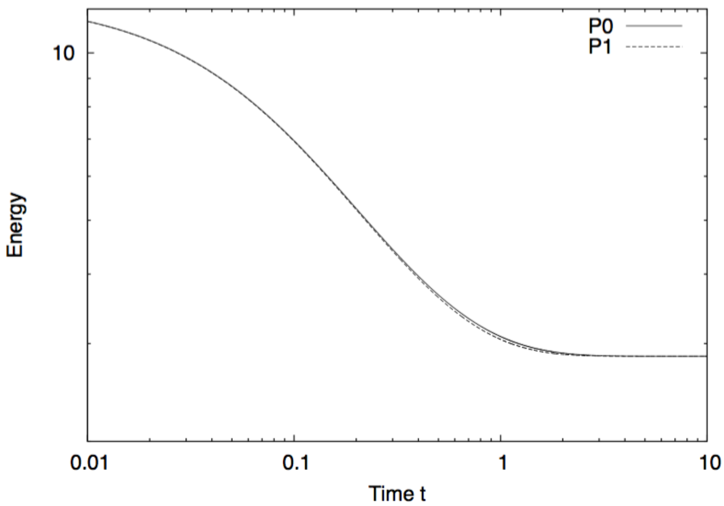

Figure 5: Evolution of the total energy versus time, $\alpha = 1$. Due to the strong relaxation, the anisotropic part does not sensibly modifies the energy, in comparison with the Kompaneets equation.

of the P_0 model, some photons are concentrated at the origin in a first time (one must zoom close to the origin for the two intermediate pictures).

This can be explained by studying the first order moment V, see Figure 7. Indeed, we see a competition between the thermalization term (Fokker-Planck term) and the friction term $-\alpha V$.

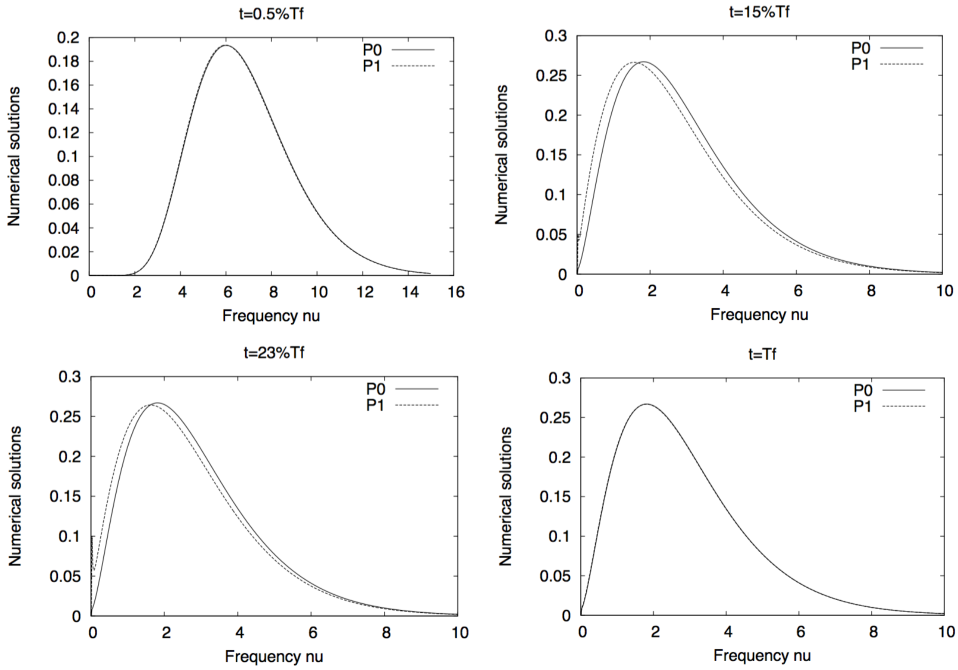

Figure 6: Zero-th order moments (P_0 and P_1) versus frequency ν at different times, $\alpha = 0.1$. A significant difference between the P_0 and P_1 models appears in a transitional regime. In long time, the P_0 and P_1 converge toward the same Planck distribution.

In long time, the friction takes over for the thermalization, and one gets $\lim_{t\to\infty} V = 0$, and the P_1 model then reduces to the Kompaneets equation. This explains that one obtains $\lim_{t\to\infty} U = f$, where f is the solution of the Kompaneets equation. This transitional regime explains the significant difference in the time evolution of the total energy between the P_0 and P_1 model displayed in Figure 8.

4.4.5 No friction $\alpha = 0$

We perform the same study but with $\alpha = 0$. Since the initial conditions of Figure 2 satisfy the hypothesis of Theorem 14, we expect to observe the convergence in long time of the zero-th order moment U toward a Planck distribution plus a Dirac at the frequency $\nu = 0$. As for the previous numerical tests ($\alpha = 1$ and 0.1), we study the evolution, as time goes on, of the zero-th order moment U, compared to the solution of the Kompaneets equation, and the first order moment V.

Figure 9 displays the evolution of the solution of the Kompaneets equation (P_0) and the zero-th order moment of the P_1 model. It shows the expected result, that is the convergence of the zero-th order moment of the P_1 model toward a Dirac function plus a Planck distribution.

In the same way, Figure 10 shows the convergence of the first order moment of the P_1 model toward a Dirac function plus a Planck function.

Finally, Figure 11 shows the time evolution of the total energy for both the Kompaneets equation and the P_1 model. It shows in particular that in this limit case $\alpha = 0$, the introduction of an anisotropic part in the radiation modifies the stationary state in long time range. Indeed the conditions of a concentration of photons near the origin are changed by the anisotropy.

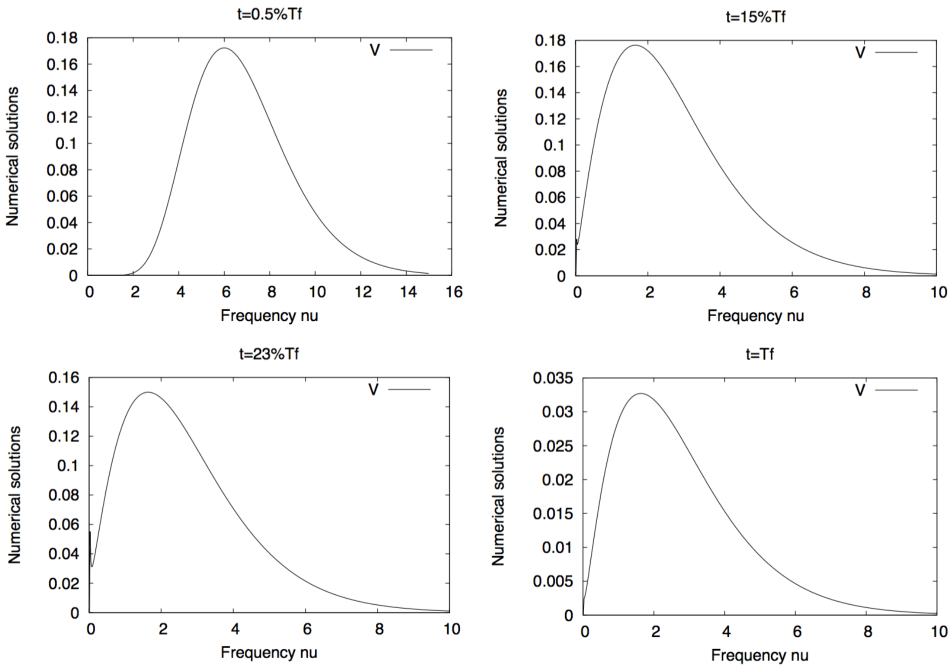

Figure 7: First order moment V versus frequency ν at different times, $\alpha = 0.1$. In short time range, there is a small concentration near $\nu = 0$. In long time the first order moment V tends to zero.

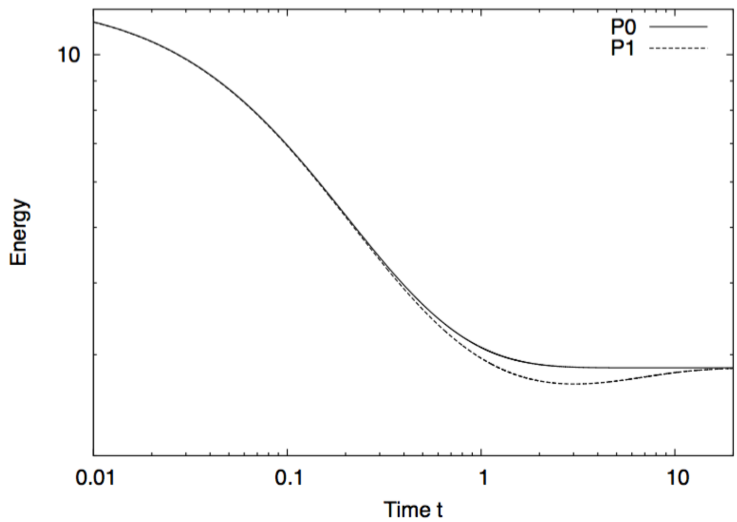

Figure 8: Evolution of the total energy versus time, $\alpha = 0.1$. The significant modification observed in the transitional regimes between the P_0 and P_1 models leads to a decrease of energy for the P_1 model.

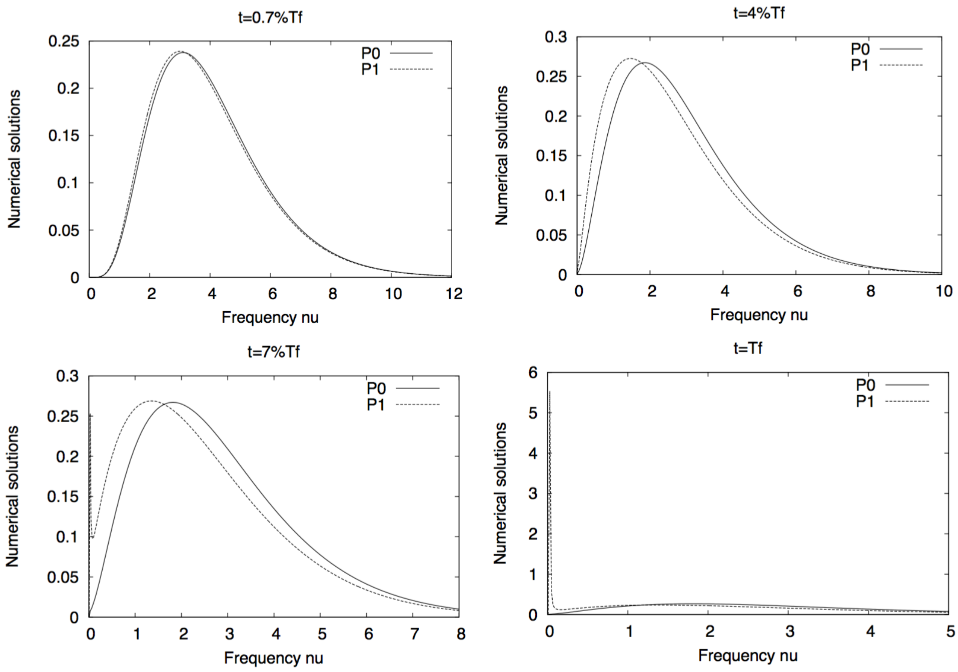

Figure 9: Zero-th order moments (P_0 and P_1) versus frequency ν at different times, $\alpha = 0$. The first order moment of the P_1 model converges to a different solution with respect to the Kompaneets equation. The final solution is the sum of a Planck type distribution and a concentration of photons near $\nu = 0$.

A Some theoretical results on the Kompaneets equation

A.1 Caflisch-Levermore results

Consider the Kompaneets equation with the boundary conditions

$$\nu^4 \Big(T\partial_\nu f + f + f^2 \Big) = 0 \text{ at } \nu = 0 \text{ and } \nu = \infty. \tag{A.1}$$

Denote $N(f_0)$ the number of photons associated to the Bose-Einstein distribution with $\mu = 0$, defined by

$$N(f_0) = \int_{\mathbb{R}^+} \frac{\nu^2}{e^{\nu/T} - 1} d\nu, \tag{A.2}$$

where T is the electronic temperature. More generally, define for any function f its associated number of photons by

$$N(f) = \int_{\mathbb{R}^+} \int_{S^2} f \nu^2 d\nu d\mathbf{\Omega}.$$

Caflisch and Levermore [CL86] studied and observed numerically the Bose condensation phenomena, recalled thereafter

$$\begin{aligned} &\bullet \text{ If } N(f^{in}) \le N(f_0), \text{ then } \exists \mu \ge 0 \text{ s. t. } \lim_{t\to\infty} \nu^2 f(t,.) = \nu^2 f_\mu(.). \\ &\bullet \text{ Else if } N(f^{in}) > N(f_0), \text{ then } \lim_{t\to\infty} \nu^2 f(t,.) = \nu^2 f_0(.) + \big(N(f^{in}) - N(f_0)\big)\delta_0. \end{aligned} \tag{A.3}$$

The parameter $\mu > 0$ in the proof is such that $N(f^{in}) = N(f_\mu)$. This result is referred to the Bose condensation phenomena; when the number of photons at the initial time is larger than a critical

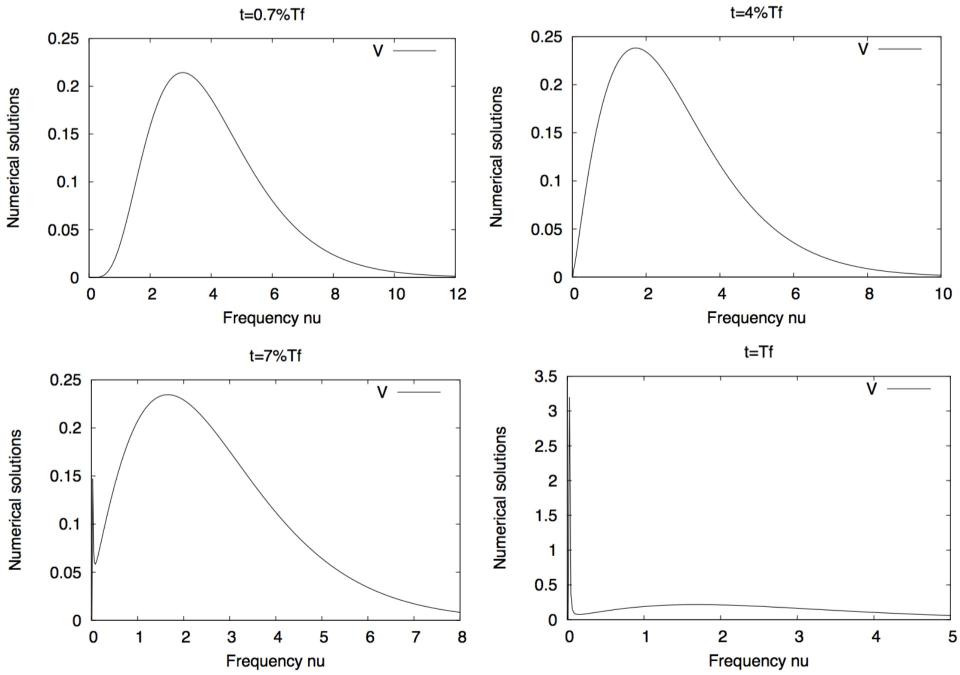

Figure 10: First order moment V versus frequency ν at different times, $\alpha = 0$. The first order moment converges toward the sum of a Planck type distribution and a concentration near $\nu = 0$.

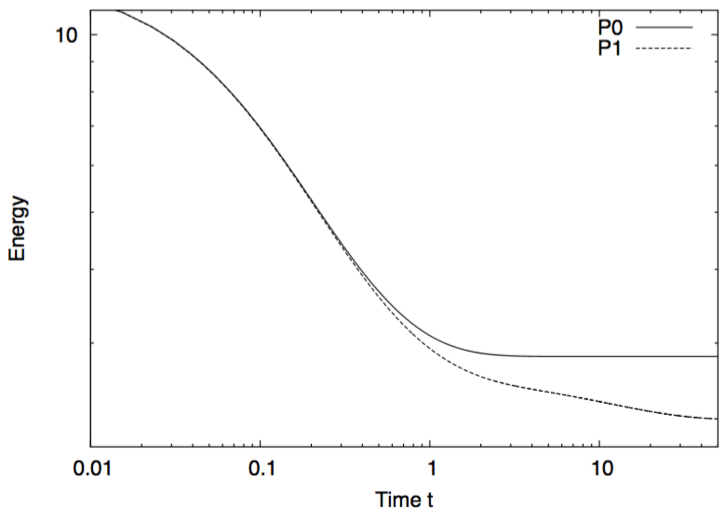

Figure 11: Evolution of the total energy versus time, $\alpha = 0$. There is a substantial decrease of the energy of the P_1 model in comparison with the Kompaneets equation, and the convergence of the numerical solution toward a different stationary solution.

number, the excess of photons is concentrated at the origin. This is illustrated in picture 13 with numerical results obtained with the method discussed in Section 4.4. Additional numerical results concerning the apparition of Dirac masses can be found in [ST95, ST97].

The Kompaneets equation is numerically solved for two different initial datum (picture 12):

the first one satisfies $N(f^{in}) < N(f_0)$, the second one $N(f^{in}) > N(f_0)$. The curves represent the

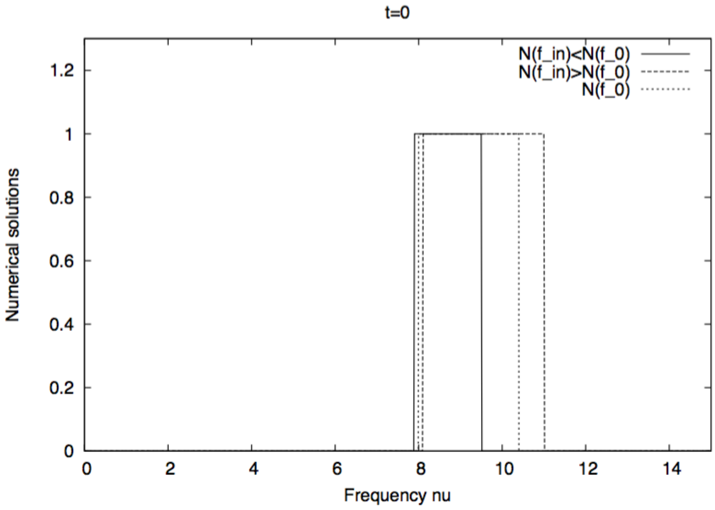

Figure 12: Initial conditions for the Kompaneets equation (4.8), written as $\nu^2 f$

quantity ν^2 multiplied by the distribution function (and not the distribution function itself).

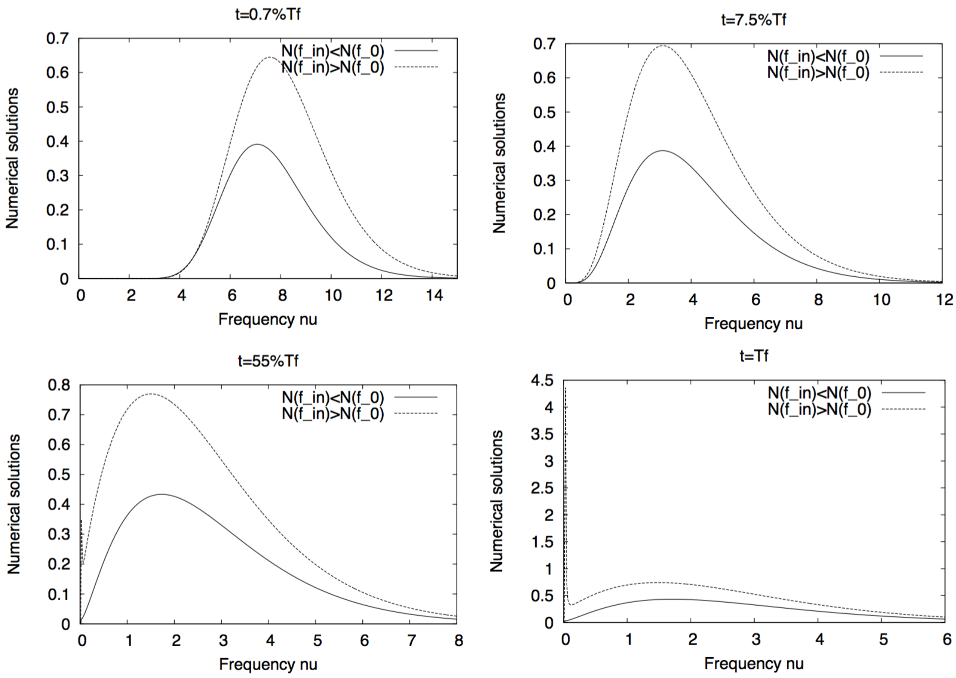

Figure 13: Numerical solutions of the Kompaneets equation (4.8) versus frequency ν. Convergence to a Planck function (full line) in the case $N(f^{in}) < N(f_0)$, and to a Planck function plus a Dirac function concentrated near $\nu = 0$ (dashed line) in the case $N(f^{in}) > N(f_0)$.

A.2 Escobedo-Herrero-Velazquez results

The condensation result of Caflish-Levermore was proved in [CL86] without discussions about the flux conditions (A.1). It must be mentioned that Escobedo-Herrero-Velazquez proved [EMV98] that there exists smooth nonnegative solutions of the Kompaneets equation, with arbitrarily small values of $N(f^{in})$, that may develop singularities near $\nu = 0$ in finite time, so that the flux condition (A.1) at $\nu = 0$ is lost. They also proved that if one replaces the flux condition at $\nu = 0$ (A.1) by an estimate of the form

$$0 \leq f(\nu, T) \leq \frac{C}{\nu^2}, \text{ as } \nu \to 0, \ C > 0,$$

then the corresponding modified problem has a unique solution for all times $t > 0$.

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